Analysis of Carr and Lee’s Quadratic Variation Derivatives Framework

Peter Larkin
Kellogg College
University of Oxford

A thesis submitted in partial fulfillment of the MSc in
Mathematical Finance
April 18, 2012
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1 Introduction</strong></td>
<td>2</td>
</tr>
<tr>
<td>2 Products traded in the market and a derivation of the price of a</td>
<td></td>
</tr>
<tr>
<td>Variance Swap</td>
<td>4</td>
</tr>
<tr>
<td>2.1 Reasons for trading volatility</td>
<td>4</td>
</tr>
<tr>
<td>2.2 Volatility Swaps</td>
<td>5</td>
</tr>
<tr>
<td>2.3 Forward Volatility Agreements</td>
<td>5</td>
</tr>
<tr>
<td>2.4 VIX</td>
<td>5</td>
</tr>
<tr>
<td>2.5 Variance Swaps</td>
<td>6</td>
</tr>
<tr>
<td>2.6 Using the information in Option Prices</td>
<td>7</td>
</tr>
<tr>
<td>2.6.1 Local volatility from Option Prices</td>
<td>7</td>
</tr>
<tr>
<td>2.6.2 Log-contract Pricing and Hedging</td>
<td>8</td>
</tr>
<tr>
<td>2.7 Constant Elastic Volatility (CEV)</td>
<td>11</td>
</tr>
<tr>
<td><strong>3 The work of Carr and Lee</strong></td>
<td>13</td>
</tr>
<tr>
<td>3.1 Futures price dynamics</td>
<td>13</td>
</tr>
<tr>
<td>3.2 Stochastic Clock</td>
<td>14</td>
</tr>
<tr>
<td>3.2.1 Measuring the clock</td>
<td>14</td>
</tr>
<tr>
<td>3.3 Skew Function</td>
<td>15</td>
</tr>
<tr>
<td>3.4 Effective Risk Neutral Process of Instantaneous Volatility</td>
<td>16</td>
</tr>
<tr>
<td><strong>4 CEV model example of the Carr and Lee framework</strong></td>
<td>17</td>
</tr>
<tr>
<td>4.1 CEV illustration of Carr and Lee work – stochastic clock with</td>
<td>18</td>
</tr>
<tr>
<td>current and terminal values known ex-ante</td>
<td></td>
</tr>
<tr>
<td>4.2 Martingale Representations of the clock</td>
<td>22</td>
</tr>
<tr>
<td>4.3 Pricing Quadratic variation derivatives – stochastic clock terminal</td>
<td>26</td>
</tr>
<tr>
<td>value unknown, using stochastic clock probability density function</td>
<td></td>
</tr>
</tbody>
</table>
5 Numerical Methods
  5.1 Finite difference ........................................... 29
  5.2 Application to options on quadratic variation with skew .... 31
  5.3 Stability Analysis ........................................... 32
  5.4 Numerical solutions of PDE (6.1) .............................. 33

6 Conclusion .......................................................... 37

A Appendix ............................................................. 39
  A.1 Log contract Pricing – alternative derivation ................. 40
  A.2 Matlab code for fair volatility/variance swap pricing ...... 42
  A.3 Bessel Functions ............................................... 45
    A.3.1 Modified Bessel functions ................................. 45
    A.3.2 Modified Bessel Solutions ................................. 45
  A.4 Toy examples .................................................. 48
  A.5 Explicit Finite Difference Matlab implementation of (6.1) .... 50
  A.6 Instability at small quadratic variation points ............... 52
  A.7 Relation between density of process and option prices ....... 53
List of Figures

2.1 Future variance quotes for equity index KOSPI2 available from Bloomberg. 6
2.2 Implied volatility surface of FTSE 100 from Bloomberg dated 06/04/2012 12

5.1 Solution of PDE (6.1) at $\tau = 0$ with pay-off $\min(q, 10), \beta = -1/2$. 33
5.2 Solution of PDE (6.1) at $\tau = T$ with pay-off $\min(q, 10), \beta = -1/2$. 34
5.3 Solution of PDE (6.1) at $\tau = 0$ with pay-off $\min(q, 10), \beta = 0$.$\ldots$ 34
5.4 Solution of PDE (6.1) at $\tau = T$ with pay-off $\sqrt{q}, \beta = -1/2$. 35
5.5 Solution of PDE (6.1) from $\tau = 0$ to $\tau = T$ to with pay-off $\sqrt{q}, \beta = -1/2$. 35
5.6 Solution of PDE (6.1) from $\tau = 0$ to $\tau = T$ to with pay-off (with High fixed Q) $\max(X, 0), \beta = -1/2$. 35
5.7 Solution of PDE (6.1) from $\tau = 0$ to $\tau = T$ to with pay-off (with Low fixed Q) $\max(X, 0), \beta = -1/2$. 36

A.1 Modified Bessel function of the first kind, $I_\alpha(x)$. 46
A.2 Modified Bessel function of the second kind, $K_\alpha(x)$. 46
A.3 Modified Bessel function solution, $x$ dependence $e^{x/2}K_\alpha(e^{x/2})$. 47
A.4 Modified Bessel function solution, $x$ dependence $e^{x/2}I_\alpha(e^{x/2})$. 47
A.5 Solution of PDE (6.1) at $\tau = 0$ with pay-off $\max(X, 0), \beta = -1/2$. 52
Acknowledgements

Firstly, I would like to thank Dr. Peter Carr for agreeing to supervise this thesis and for giving up many hours of his time between March 2011 and June 2011. The meetings at Morgan Stanley in New York were of great use and were where most of the ideas contained here arose from. My gratitude must also be given to Dr. Jeff Dewynne and Dr. Christoph Reisinger for helpful email exchanges about the numerical solution of PDE (6.1). Thanks also to my employer, d-fine Ltd for giving me the opportunity to take the part-time MSc in Mathematical Finance at Oxford. Last but not least, I would like to thank my partner Shanti for her support during the writing of the thesis.
Chapter 1

Introduction

Over the last years, there has been a growing interest in pricing and hedging financial products contingent on the volatility or variance of tradable assets [53, 2, 4, 38, 73, 13, 15, 37, 39, 40, 11, 43, 48, 49, 50, 54, 55, 56, 57, 58, 60]. In parallel to this, there is a fundamental need to price in such a way as to capture all the information available in the market – in particular, in the observed implied volatility smile\(^1\) [7, 8, 9, 67, 68, 16, 41].

The volatility of an equity is the simplest measure of how risky it is, or perhaps how much it is likely to move around in the future, based on how it has moved historically, or what the market implies it to be in the future. Investors may wish to trade volatility if they believe they have some insight into the level of future volatility. For example, if a trader thinks that volatility is currently too low, he or she may want to take a position which allows them to profit if volatility increases [60].

In this work we are interested in a one important part of this growing area – the pricing and hedging of European options whose pay-off at maturity depends on the quadratic variation of the underlying process [18, 19, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. The (discrete) quadratic variation of a stochastic process \(X_t\) is defined as [20]

\[
q(X_t) = \mathbb{E}^Q \left[ \sum_{i=1}^{N} \left[ \log \left( \frac{X_{t_i}}{X_{t_{i-1}}} \right) \right]^2 \right],
\]

(1.1)

where \(\mathbb{E}^Q\) is the expectation under the risk-neutral measure and \(X_t\) is the value of some process at time \(t\). If we make \(N\) discrete observations over the life of the contract

\(^1\)We can think of the smile arising from the fact that in reality the distribution which equities is not log-normal but has fatter tails meaning that large deviations are more common leading to a larger price of options which would benefit from larger movements.
with maturity $T$, the annualised quadratic variation of the stochastic process is then

$$
\bar{q}(X_t) = \mathbb{E}^Q \left[ \frac{k}{N} \sum_{i=1}^{N} \left[ \log \left( \frac{X_{t_i}}{X_{t_{i-1}}} \right) \right]^2 \right] = \frac{k}{N} q(X_t),
$$

(1.2)

where $k$ is the number of observations per year. Quadratic variation is often used as a measure of realised variance, since this is the quantity really observed in the market, and moreover, for models without jumps, the quadratic variation is the same as the definition of realised variance. For a continuous stochastic process, for example:

$$
\frac{dX_t}{X_t} = \mu dt + \sigma_t dW_t,
$$

(1.3)

the quadratic variation is:

$$
q(X_t) = \int_0^t \sigma_s ds.
$$

(1.4)

There have been a number of papers in recent years on the pricing of derivatives contingent on this quantity [53, 2, 4, 38, 67, 68, 73, 13, 15, 16, 37, 39, 40, 41, 11, 43, 48, 49, 50, 54, 55, 56, 57, 58, 60, 33, 34]. We will be paying particular attention to the recent work of Carr and Lee [18, 25, 26] who have constructed a framework for the pricing of volatility derivatives based on just a few simple and intuitive assumptions and incorporate the observed skew/smile observed in the market. They show how to hedge and price a claim paying a nonlinear function of the terminal log futures price and its terminal quadratic variation. Though their work is quite general, we will look at specific examples related to the Constant Elastic Volatility (CEV) model [50, 42] which is contained within their framework. We will introduce background material as necessary to explain the examples we will illustrate and to explain any new material.
Chapter 2

Products traded in the market and a derivation of the price of a Variance Swap

As a motivation for the reader, we will briefly explain some of the most common products which are traded to hedge or take a view on volatility, and we will go into depth on perhaps the most liquid such product – Variance Swap, and show the common market model used to derive its fair price.

2.1 Reasons for trading volatility

Volatility is likely to increase when uncertainty and risk grow. When volatilities are high, they typically decrease with time and vice-versa, showing they have a mean-reverting behaviour analogous to interest rates. Volatility of an equity is often negatively correlated to the equity price changes [71]. These characteristics draw traders to invest in volatility products [35]:

- If a trader wishes to speculate about the future levels of volatility of a stock or an index, he/she can go long or short realised volatility with a swap.

- The trader might also trade the spread between realised and implied volatility.

- Volatility tends to increase during global equity declines, and so a trader in an equity fund might short volatility due to this negative correlation between index level and its volatility on the expectation that volatility will increase – this has been especially prevalent since the 2008 global financial crisis [74].
2.2 Volatility Swaps

The most direct way to take a position in volatility, is to use volatility swaps\(^1\) since they provide the trader with a pure exposure to volatility. A stock volatility swap is a forward contract on annualised volatility with pay-off equal to:

\[
(\sigma_R - K_{vol}) \times N, \tag{2.1}
\]

where \(\sigma_R\) is the realised volatility of the stock over the life of the contract\(^2\), \(K_{vol}\) is the annualised volatility delivery price and \(N\) is the notional amount of the swap in a given currency per annualised volatility point. In other words, the holder of the volatility swap receives at expiry, \(N\) for every point by which the stock’s realised volatility is above the delivery price; the holder is swapping a fixed volatility\(^3\) for a floating one. As usual with swaps, the \(K_{vol}\) is chosen to make the swap have zero value at initiation. There could in principle be several payments on the fixed leg of the contract, but as long as the initial contract is traded at par – valuation will be the same on the (floating) volatility leg. There are a couple of conventions to calculate the realised standard deviation, either by subtracting the means from each return, or by assuming a zero mean. The latter case is often preferred since this contract can be replicated using a series of options, which we will discuss later.

2.3 Forward Volatility Agreements

A forward volatility agreement works in a similar way to other forward contracts, but is on the realised or implied volatility of the returns of some underlying reference entity. The pay-off at expiry of based on the difference between the pre-specified volatility level (determined at trade inception) and the volatility level at the expiry or settlement date in the future.

2.4 VIX

The VIX – volatility index shows the markets expectations of the next 30-days volatility based on the weighted implied volatilities of the S&P 500 index options – both call and put. This is used as a measure of market risk. There are other similar volatility indices like the VNX which is related to the NASDAQ 100 and the VXD which is

\(^1\)sometimes called realized volatility forward contracts
\(^2\)quoted as an annual amount
\(^3\)\(K_{vol}\) is quoted as a percentage
related to the Dow Jones Industrial Average. For more information on the pricing of VIX options and futures, see [36, 72] and the references there in.

2.5 Variance Swaps

A variance swap is a forward contract on annualised variance, which is the square of the realised volatility [35]. The payoff at expiry is

\[(\sigma^2_R - K_{var}) \times N,\]  

(2.2)

where \(K_{var}\) is the delivery price for variance and \(N\) is quoted in a unit of currency per annualised volatility point squared. This gives the holder of the swap at expiration \(N\) units of currency for every point \(\sigma^2_R - K_{var}\) exceeds zero. As usual, the fair value of a variance swap is the delivery price which makes the swap have zero value. As a concrete example, imagine we have a variance swap on the South Korean KOSPI2 Index.

Trading systems at financial institutions will need to save the historical settlement prices for each business day for the KOSPI2 Index, as we need these going back to
the start date of the trade in order to calculate the realized variance – the number of fixings will be one less than the number of observations.

\[ \sigma^2_{released} = \frac{252}{N} \sum_{i=1}^{N} \left( \frac{S_{i+1}}{S_i} \right)^2, \]

where 252 converts the variance to an annualised rate using the convention of 252 business days per calendar year. \( N \) is the number of fixings, for example, if we have settlement prices for 25 banking days in the past, at the end of the 25th day, we can compute 24 quotients, take the log and square, sum up, divide by 24, and multiple 252.

Typically, to get the forward rate, smaller financial institutions will get quotes from Bloomberg (see figure (2.5)) from one of the major investment banks which quote future volatility or variance estimates at certain future expiries. Then the future variance value is an interpolation of these values to the expiry of the trade. The most common mathematical way to obtain the future expected variance is using a set of call and put option prices which we will describe in section (2.6.2).

### 2.6 Using the information in Option Prices

#### 2.6.1 Local volatility from Option Prices

In the presence of volatility smiles/skews, we need to be able to price options taking into account this phenomenon [71]. By this, we mean we can obtain a formula for \( \sigma_{K,T} \) – the volatility as a function of strike \( K \), maturity \( T \) from option prices. The forward Fokker-Planck equation for option prices is

\[
\left( \frac{\partial}{\partial T} + (r - d)K \frac{\partial}{\partial K} - \frac{1}{2} \sigma_{K,T}^2 K^2 \frac{\partial^2}{\partial K^2} + d \right) C_{K,T}(0, S) = 0,
\]

where \( C_{K,T}(t, S) \) is the plain vanilla (call) option price at time \( t \), and \( d \) is the dividend, \( r \) is the risk-free rate, \( K \) the strike, and \( T \) the maturity of the option. Rearranging this, we have

\[
\sigma_{K,T}^2 = 2 \frac{\partial^2 C_{K,T}(0, S)}{\partial K^2} + (r - d)K \frac{\partial C_{K,T}(0, S)}{\partial K} + dC_{K,T}(0, S).
\]

This is the local volatility that we imply to be obtained at time \( T \), given today’s option prices, when the future stock price is equal to \( K \). The second derivative of

---

\(^4\text{Sometimes referred to as Bus/252 convention, c.f. Act/365}\)
the call option price with respect to strike which appears is actually the density of the process – see appendix (A.7) for a derivation of this. When we use this volatility in the Black-Scholes formula to obtain option prices, it is often remarked that we are using the wrong number in the wrong equation to obtain the right price. This approach is used since we observe in the market that the volatility has a dependence on maturity and strike, and we can obtain this dependence from option prices. In the next section, we will analogously see how we can obtain the future expected variance from todays’ option prices, which can then be used to price Variance Swaps.

2.6.2 Log-contract Pricing and Hedging

If an investor would like a long position in future realised variance, a single option is not the perfect product since as soon as the stock price changes, the sensitivity to future changes in variance is altered. An investor really would like a portfolio whose sensitivity to realised variance is independent of the stock. We will see that we are able to achieve this by trading in options of many strikes; we show that if we have a combination of options inversely weighted in the strike squared $K^2$, the realised variance will be independent of the stock price, [53, 35, 38].

We will show that we can replicate the log contract pay-off with Bonds, Futures and a continuum of vanilla options [38] over all strikes; in practice, the integrals will be replaced with a discrete summation over all strikes available. This framework of replicating the pay-off in terms of bonds, futures and options will be drawn upon several times in this work. By the no-arbitrage principle, this replication is equivalent to hedging\(^5\).

We will derive this replication here and in appendix (A.1) show a related equivalent derivation. We start with Itô’s lemma and we assume that the underlying dynamics is described by

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{2.3}
\]

For our arguments, we’ll see that it is not necessary to move to the risk-free measure, since this just changes the drift and not the volatility/variance that we are interested in. Itô gives us

\[
d(\log S_t) = (\mu - \frac{\sigma^2}{2}) dt + \sigma dW_t
\]

\[
\frac{dS_t}{S_t} - d(\log S_t) = \frac{\sigma^2}{2} dt. \tag{2.4}
\]

\(^5\)albeit with a sign change
Integrating both sides, give the total variance to be
\[
Var = \frac{1}{T} \int_0^T \sigma^2 dt = \frac{2}{T} \left( \int_0^T \frac{dS_t}{S_t} - \log \left( \frac{S_T}{S_0} \right) \right).
\] (2.5)

This equation reads that to match the total variance between now and expiry \( T \), we need to continuously hedge \( \frac{1}{S_t} \) and short a log contract. Using a similar argument to above, or using that for an payoff \( g(G) \), the sifting\(^6\) property of the Dirac delta function gives us
\[
g(F) = \int_0^\infty g(K) \delta(G - K) dK
\]
\[
= \int_0^\kappa g(K) \delta(G - K) dK + \int_\kappa^\infty g(K) \delta(G - K) dK,
\] (2.6)

for any non-negative \( \kappa \). Integrating by parts twice, we have
\[
g(G) = g(K)1_{(G<\kappa)}\bigg|_0^\kappa - \int_0^\kappa g'(K)1_{(G<\kappa)} dK +
\]
\[
g(K)1_{(G\geq\kappa)}\bigg|_\kappa^\infty + \int_\kappa^\infty g'(K)1_{(G\geq\kappa)} dK
\]
\[
= g(\kappa)1_{(G<\kappa)} - g'(K)(G - K)^+\bigg|_0^\kappa + \int_0^\kappa g''(K)(G - K)^+ dK +
\]
\[
g(\kappa)1_{(F\geq\kappa)} - g'(K)(G - K)^+\bigg|_\kappa^\infty + \int_\kappa^\infty g''(K)(G - K)^+ dK
\]
\[
= g(\kappa) + g'(\kappa)[(G - \kappa)^+ - (\kappa - G)^+] + \int_0^\kappa g''(K)(G - K)^+ dK + \int_\kappa^\infty g''(K)(G - K)^+ dK.
\] (2.7)

Taking the log contract, we have, up to an arbitrary constant \( S^* \)
\[
- \log \left( \frac{S_T}{S^*} \right) = \frac{S_T - S^*}{S^*} + \int_{K \leq S^*} (K - S_T)^+ \frac{dK}{K^2} + \int_{K \geq S^*} (S_T - K)^+ \frac{dK}{K^2}.
\] (2.8)

This decomposition of the log contract can be read as a short position in \( 1/S^* \) forward contracts struck at \( S^* \), a long position in \( 1/K^2 \) put options with strike \( K \), for all \( K \) from 0 to \( S^* \) and a long position likewise in call options. This means that we have for the variance
\[
Var = \frac{2}{T} \left( rT - \left( \frac{S_0}{S^*} e^{rT} - 1 \right) - \log \frac{S^*}{S_0} + e^{rT} \int_0^{S^*} \frac{1}{K^2} P(K) dK + e^{rT} \int_{S^*}^\infty \frac{1}{K^2} C(K) dK \right).
\] (2.9)

\(^6\)Sometimes also referred to as the sampling property
Taking $S^*$ to be the current forward price $S^* = F_T = S_0 e^{rT}$, the fair variance swap strike (often denoted $K_{var}$) can be written as

$$Var = K_{var} = \frac{2e^{rT}}{T} \left( \int_0^F \frac{1}{K^2} P(K) dK + \int_F^\infty \frac{1}{K^2} C(K) dK \right).$$

(2.10)

We provide an implementation of this in Matlab (see attached Matlab file var-SwapOnFTSE100ForThesis.m) for future variance estimates of the FTSE 100. The model [39] 2.10 take as inputs, the spot, and a set of option prices. We obtain the option prices from the implied volatility surface from Bloomberg for FTSE 100 (see figure (2.6.2) which also shows us the skew and smile characteristics of the implied volatility surfaces.), and the 3 month GBP LIBOR curve which we’ll take as an appropriate risk-free curve. We interpolate with Matlab’s interpolation method with type ‘linear’ on the volatility surface between a set of strikes and expiries to compute the $C(K)$ and $P(K)$ prices.

Implied volatility surface:

<table>
<thead>
<tr>
<th>Strike/Expiry</th>
<th>4300</th>
<th>4600</th>
<th>4900</th>
<th>5200</th>
<th>5500</th>
<th>5800</th>
<th>6100</th>
<th>6400</th>
<th>6700</th>
<th>7000</th>
</tr>
</thead>
<tbody>
<tr>
<td>20/04/2012</td>
<td>0.576</td>
<td>0.487</td>
<td>0.399</td>
<td>0.311</td>
<td>0.230</td>
<td>0.172</td>
<td>0.182</td>
<td>0.221</td>
<td>0.259</td>
<td>0.296</td>
</tr>
<tr>
<td>15/06/2012</td>
<td>0.364</td>
<td>0.320</td>
<td>0.277</td>
<td>0.236</td>
<td>0.198</td>
<td>0.164</td>
<td>0.144</td>
<td>0.146</td>
<td>0.156</td>
<td>0.171</td>
</tr>
<tr>
<td>21/12/2012</td>
<td>0.286</td>
<td>0.263</td>
<td>0.240</td>
<td>0.217</td>
<td>0.196</td>
<td>0.176</td>
<td>0.158</td>
<td>0.145</td>
<td>0.137</td>
<td>0.135</td>
</tr>
<tr>
<td>21/06/2013</td>
<td>0.277</td>
<td>0.257</td>
<td>0.239</td>
<td>0.221</td>
<td>0.204</td>
<td>0.188</td>
<td>0.174</td>
<td>0.162</td>
<td>0.153</td>
<td>0.147</td>
</tr>
<tr>
<td>20/12/2013</td>
<td>0.271</td>
<td>0.254</td>
<td>0.238</td>
<td>0.223</td>
<td>0.209</td>
<td>0.196</td>
<td>0.183</td>
<td>0.173</td>
<td>0.164</td>
<td>0.157</td>
</tr>
</tbody>
</table>

The risk-free curve can be found in attached excel sheet VarianceSwapCalcInputData.xls. We obtain the following fair future volatility values:

<table>
<thead>
<tr>
<th>Date</th>
<th>Fair Vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>20/04/2012</td>
<td>20.7994</td>
</tr>
<tr>
<td>15/21/2012</td>
<td>19.7335</td>
</tr>
<tr>
<td>21/12/2012</td>
<td>19.1331</td>
</tr>
<tr>
<td>21/06/2013</td>
<td>19.1067</td>
</tr>
<tr>
<td>20/12/2013</td>
<td>18.8076</td>
</tr>
</tbody>
</table>

We see that these are generally decreasing with time which is in line with the general negative slope towards expiry of the implied volatility surface.

One can derive forward volatilities from these calculated values, for example, if one entered into a volatility or variance swap which starts in 21/06/2013 and ends 20/12/2013, we could use the fact that variances are additive to obtain the expression

$$\sigma_{t,T} = \sqrt{(T - t_0)\sigma_{t_0,T}^2 - (t - t_0)\sigma_{t_0,t}^2} / (T - t),$$

(2.11)
and use this to determine the forward volatility. We will now finish off this introductory chapter reviewing what defines a CEV process \[42\] since this will be mentioned several times later.

### 2.7 Constant Elastic Volatility (CEV)

Constant Elastic Volatility (CEV) processes take the general form \[42\]

\[
dX = a(X, t)dt + \sigma(\beta)X^{\beta+1}dW, \tag{2.12}
\]

such that \(X(0) > 0\). There are variations of this where instead of \(\beta + 1\) in the exponent, \(\beta\) is used. We introduce the additional 1 since it will simplify notation later when we consider the process \(dX/X\). One problem with this process is that it doesn’t satisfy the local Lifschitz condition\(^7\) for \(-1 \leq \beta \leq 0\). In spite of this, the process (for \(-1 \leq \beta \leq 0\)) does have the following behaviour:

- for \(\beta \geq -1/2\), (2.12) has a unique solution
- for \(\beta = 0\), the process cannot reach \(x = 0\) (c.f Geometric Brownian Motion)
- for \(-1 \leq \beta \leq 0\), the process can reach \(x = 0\)
- for \(-1 \leq \beta \leq -1/2\), (2.12) doesn’t possess a unique solution, unless we use a separate boundary condition at \(x = 0\)

We note that if we were to allow \(\sigma\) to be stochastic itself, we would have something more like a SABR model \[70\]. In the work below, the stochastic nature of the volatility is pushed onto the \(dW\) to become \(dW_\tau\) where \(\tau\) is a stochastic clock which the Brownian motion runs on – we will explain in more detail later.

\(^7\)For any \(\epsilon > 0\), one can always find a constant \(C_\epsilon\) such that, for \(x\) and \(y\) positive but smaller than \(\epsilon\), \(|\phi(x) - \phi(y)| \leq C_\epsilon(x - y)|\)

\(^8\)From the average value theorem, i.e given any differentiable function \(\phi()\), and two points in its domain, \(x\) and \(y\), it should always be possible to write \(\frac{\phi(x) - \phi(y)}{x - y} = \phi'(\eta)\) for some \(\eta \in [x, y]\). We have \(\phi(x) = x^{\beta+1}\), so \(\phi'(x) = (\beta + 1)x^\beta\) which is unbounded for \(-1 < \beta < 0\)

11
Figure 2.2: Implied volatility surface of FTSE 100 from Bloomberg dated 06/04/2012
Chapter 3

The work of Carr and Lee

Carr and Lee [20] have come up with a general framework for pricing and hedging European options which pays a nonlinear function of the quadratic variation of the log futures price at maturity $T$ and the terminal log futures price. We proceed by reviewing their general framework and later, we will illustrate parts of the framework for the specific example of the CEV case [42]. We follow the assumptions of Carr and Lee [20] which we will also summarise for the readers convenience. Let $F_t$ denote the filtration at $t \in [0, T]$.

- Frictionless markets i.e. no transaction costs are associated with buying or selling financial products.

- Continuous trading is available in Riskless Bonds, Futures on a risky asset and Vanilla Options on these futures of all strikes, all with same maturity, and there are no arbitrage opportunities between the three sets of assets.

- The filtration $\mathcal{F}_t$ includes the path over $[0, t]$ followed by the market prices of the riskless bond, the futures price and European options of all positive strikes.

- For simplicity, we assume zero interest rates, but the model is extendible with ease to at least deterministic interest rates.

3.1 Futures price dynamics

Let $F$, the futures price of a risky asset be a continuous positive stochastic process under the real-world statistical probability measure $\mathbb{P}$. By the no-arbitrage principle, there exists a risk-neutral probability measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$ such that $F$ is a continuous positive martingale [75]. We define $W$ to be a $\mathbb{Q}$-standard Brownian motion which drives the futures prices. This Brownian motion will be time-changed to
run on the stochastic clock defined below. Then \( F \) solves the following time-changed stochastic differential equation (SDE):

\[
d\frac{F_t}{F_t} = s \left( \log \left( \frac{F_t}{F_0} \right) \right) d\tau_t, \quad t \in [0,T],
\]

(3.1)

where \( F_0 \) is the Futures price at \( t = 0 \), and \( s(x) \) is what we refer to as skew function. It is also an important assumption that \( \alpha \) evolves independently of \( W \).

### 3.2 Stochastic Clock

We can consider time-changed Brownian motion [45], such that the Brownian motion runs on a clock whose rate depends on the speed in market activity. This speed can be thought of as being contingent on the arrival of new news and increases during periods of intense trading. We define an additive process \( \tau \) by:

\[
\tau_t \equiv \int_0^t \alpha_{s}^2 ds, \quad t \in [0,T],
\]

(3.2)

where \( \alpha \) is a real-valued stochastic process. As \( \tau \) is increasing, and it starts at zero, we call it a Stochastic clock [46, 61, 62], and we refer to \( \alpha \) as the activity rate. Note that the process \( \alpha \) can include jumps but \( \tau \) will still be continuous. Mathematically, typical dynamics for \( \alpha \) could be a mean-reverting process like the Ornstein-Uhlenbeck process [63]

\[
d\alpha_t = \theta(\mu - \alpha_t)dt + \sigma_\alpha dW_t.
\]

(3.3)

Running the Brownian motion on this stochastic clock has the affect of introducing a stochastic volatility into the model, but will mean we need to use different machinery in order to measure and calibrate it.

#### 3.2.1 Measuring the clock

Following the work of Carr and Lee [20], we show that \( \tau_t \) can be inferred from the log futures price preceding it. By taking the square of (3.1), we obtain the quadratic variation (QV) of the futures price process:

\[
\frac{d \langle F \rangle_t}{F_t^2} = s^2 \left( \log \left( \frac{F_t}{F_0} \right) \right) d\tau_t
\]

\[
= s^2 \left( \log \left( \frac{F_t}{F_0} \right) \right) \alpha_{t}^2 dt, \quad t \in [0,T]
\]

(3.4)
Let \( Y_t \equiv \log \left( \frac{F_t}{F_0} \right) \) over the time period \([0, t]\) where \( Y_0 = 0 \). We now use Itô’s formula:

\[
dY_t = \frac{dF_t}{F_t} - \frac{d\langle F \rangle_t}{2F_t^2} = \frac{dF_t}{F_t} - \frac{s^2(Y_t)}{2} \alpha_t^2 dt, \quad t \in [0, T].
\]

(3.5)

Taking the quadratic variation of both sides, we obtain:

\[
d\langle Y \rangle_t = \frac{d\langle F \rangle_t}{F_t^2} = s^2(Y_t) \tau_t, \quad (3.6)
\]

where \( \langle Y \rangle_t = 0 \). This means we can infer \( \tau_t \) from the path taken by \( Y \) over \([0, T]\)

\[
\tau_t = \int_0^t \frac{d\langle Y \rangle_u}{s^2(Y_u)}. \quad (3.7)
\]

We can think of this calculated definition of the stochastic clock as being equivalent to calculating the realised volatility between inception and expiry. Due to this ability to read the value of the clock from the path of the log futures price, one can in principle define contingent claims whose payoff depends on the stochastic clock reaching some time barrier. We will illustrate a toy model example of this later when the futures price process is CEV [42], both analytically using separation of variables and numerically using finite difference methods. This will form the bulk of our results.

### 3.3 Skew Function

When pricing an option, it is not satisfactory to only determine the price, but one should also be able to match the implied volatility structure at maturity [7, 8, 9] i.e. the evident skew and smile. We have some assumptions on the form of the skew function \( s(x) \) which try to capture this phenomenon

- \( s(x) \) is a positive function: \( \mathbb{R} \mapsto \mathbb{R}^+ \) which is known at the start.

- \( s(x) \) is restricted to functions which keep the futures price positive forever.

A sufficient condition for this latter assumption is \( s(x) \) should be bounded above when \( F \) is near the origin i.e. \( \lim_{x \to -\infty} s(x) < M \) for some positive finite \( M \). Carr and Lee [21] found that when \( s(x) \) is a symmetric function of \( x \), the Black implied volatility is then an even function of \( \log(K/F_0) \). To capture the observed downward sloping skew in the implied volatility coming from equity option prices, one needs to choose \( s(x) \) to be a decreasing function of \( x \). This is the only way in which we can generate this observed skew, hence \( s(x) \) is referred to as the skew function.
3.4 Effective Risk Neutral Process of Instantaneous Volatility

To bring the last few sections together, we would like to determine the form the instantaneous volatility will take in this framework. If we substitute (3.2) in (3.1) and using the Brownian scaling property\(^\text{1}\) implies that there exists a standard Brownian motion \(\hat{W}\) such that \(F\) has the same dynamics as the process \(\hat{F}\) which solves:

\[
\frac{d\hat{F}_t}{\hat{F}_t} = \sigma_t d\hat{W}_t, \quad t \in [0,T],
\]

where

\[
\sigma_t \equiv s \left( \log \frac{F_t}{F_0} \right) \alpha_t, \quad t \in [0,T].
\]

Since the process \(\alpha\) can include jumps, the volatility \(\sigma\) can also jump.

\(^1\)For every \(c > 0\) the process \(W_t = \frac{1}{\sqrt{c}} W_{ct}\) is another Wiener process. In our context, we have

\[
dW_t = \frac{1}{\sqrt{\alpha_t}} dW_{\tau_t} = \frac{1}{\alpha_t} dW_{\tau_t}.
\]
Chapter 4

CEV model example of the Carr and Lee framework

We have defined the notion of a stochastic clock which we are able to measure from the path taken by the log futures price. We can define a contingent claim which has a payoff dependent on the log futures price and its quadratic variation at maturity. We now explore the case when the maturity is defined to be the time at which the stochastic clock $\tau$ reaches a positive barrier $H$. These products would come under the general heading of timer-options [12]. We will keep as close as possible with the notation of Carr and Lee [20], but will elaborate on some of the derivations in their work.

We have $x = \log \left( \frac{F_t}{F_0} \right) = Y_t$, $q = \langle Y \rangle_t$, and $R = H - \tau_t$, with $x \in \mathbb{R}, q \in \mathbb{R}^+, R \in \mathbb{R}^+$. If and when $\tau_t = H$, the clock has reached the barrier, the contingency claim matures giving us a pay-off. Introducing $V^c(x, q, R)$, we can use Itô’s lemma and the risk-neutral pricing framework to obtain a PDE which $V^c$ needs to satisfy.

\[
\begin{align*}
\frac{dV^c(x, q, R)}{d\tau_t} &= \frac{\partial V^c}{\partial x} dY_t + \frac{1}{2} \frac{\partial^2 V^c}{\partial x^2} d\langle Y \rangle_t + \frac{\partial V^c}{\partial q} d\langle Y \rangle_t + \frac{\partial V^c}{\partial R} dR \\
&= \frac{\partial V^c}{\partial x} \left( \frac{dF_t}{F_t} - \frac{1}{2} \frac{d\langle F \rangle_t}{F_t^2} \right) + \frac{1}{2} \frac{\partial^2 V^c}{\partial x^2} s^2(Y_t) d\tau_t + \frac{\partial V^c}{\partial q} s^2(Y_t) d\tau_t - \frac{\partial V^c}{\partial R} d\tau_t \\
&= \left[ s^2(Y_t) \left( \frac{1}{2} \frac{\partial^2 V^c}{\partial x^2} - \frac{1}{2} \frac{\partial V^c}{\partial x} + \frac{\partial V^c}{\partial q} \right) \right. \left. - \frac{\partial V^c}{\partial R} \right] d\tau_t + \frac{\partial V^c}{\partial x} s^2(Y_t) dW_{\tau_t}.
\end{align*}
\]

(4.1)

Under the risk-neutral pricing framework, we require $V^c$ to be a martingale, and so the drift term should vanish. This leads to the following convection-diffusion\footnote{Terms like $\frac{\partial^2 V}{\partial x^2}$ are referred to diffusion terms, and $\frac{\partial V}{\partial x}$ convection or transportation} PDE
which needs to be solved:

\[
\begin{align*}
    s^2(x) \left( \frac{1}{2} \frac{\partial^2 V^c(x, q, R)}{\partial x^2} - \frac{1}{2} \frac{\partial V^c(x, q, R)}{\partial x} + \frac{\partial V^c(x, q, R)}{\partial q} \right) &= \frac{\partial V^c(x, q, R)}{\partial R}, \\
    \end{align*}
\]

with initial condition:

\[
V^c(x, q, 0) = h(x, q), \quad x \in \mathbb{R}, q \in \mathbb{R}^+.
\]

Carr and Lee assume the option price should satisfy PDE (6.1) to show how the payoff can be replicated. When the stochastic clock reaches the barrier, we label this time as \( t = S \), if it never reaches the barrier, we have \( S = \infty \). Integrating (4.1) between \( t = 0 \) and \( S \) and using the fact that \( d\tau_t = \alpha^2 t dt \), we have:

\[
V^c(Y_S, \langle Y \rangle_S, 0) = V^c(0, 0, H) + \int_0^S \frac{\partial V^c}{\partial x}(Y_t, \langle Y \rangle_t, H - \tau_t) dY_t + \int_0^S s^2(Y_t) \frac{\partial^2 V^c}{\partial x^2}(Y_t, \langle Y \rangle_t, H - \tau_t) - \frac{1}{2} \frac{\partial V^c}{\partial x}(Y_t, \langle Y \rangle_t, H - \tau_t) \]

\[
+ s^2(Y_t) \frac{\partial V^c}{\partial q}(Y_t, \langle Y \rangle_t, H - \tau_t) - \frac{\partial V^c}{\partial R}(Y_t, \langle Y \rangle_t, H - \tau_t) \alpha^2 dt. \quad (4.4)
\]

If \( V^c \) solves PDE (6.1), the above equation simplifies to:

\[
V^c(Y_S, \langle Y \rangle_S) = h(Y_S, \langle Y \rangle_S) = V^c(0, 0, H) + \int_0^S \frac{\partial V^c}{\partial x}(Y_t, \langle Y \rangle_t, H - \tau_t) \frac{dF_t}{F_t}, \quad (4.5)
\]

which we can naturally read off as a replication of the payoff; we can replicate the payoff \( h(Y_S, \langle Y \rangle_S) \) by charging an initial amount \( V^c(0, 0, H) \) cash and holding \( \frac{\partial V^c}{\partial x}(Y_t, \langle Y \rangle_t, H - \tau_t) \frac{dF_t}{F_t} \) futures for each \( t \in [0, S] \). We have found that we can replicate the payoff of QV claims simply with risk-free assets and futures. We haven’t needed to include a continuum of options in order to maintain the hedge. It was important though that we knew ex-ante, the current and terminal values of the stochastic clock for each \( t \in [0, S] \). We would like to generalise this result to the case when we don’t know ex-ante, these values and so then we will need to know the probability density function of the terminal clock.

### 4.1 CEV illustration of Carr and Lee work – stochastic clock with current and terminal values known ex-ante

We aim to find an analytical solution for one set of special cases of solutions of (6.1), the CEV case. For some background mathematical analysis of (6.1) and toy examples, see appendix (A.3.2)
In the CEV case, the skew function takes the form \( s(x) = \exp(\beta x) \), giving

\[
dF_t = \frac{1}{F_0^\beta} F_t^{\beta+1} dW_t. \tag{4.6}
\]

The PDE which then needs to be solved is

\[
e^{2\beta x} \left[ \frac{1}{2} \frac{\partial^2 V^c(x, q, R)}{\partial x^2} - \frac{1}{2} \frac{\partial V^c(x, q, R)}{\partial x} + \frac{\partial V^c(x, q, R)}{\partial q} \right] = \frac{\partial V^c(x, q, R)}{\partial R}, \tag{4.7}
\]

with initial condition \( V^c(x, q, 0) = h(q) \), where \( h(q) \) is the pay-off. Typical pay-offs are:

- volatility swap \( h(x, q) = \sqrt{q} \)
- capped variance swap \( h(x, q) = \min(q, k) \) where \( k > 0 \) is the positive cap.

To proceed with finding a solution, we assume separation of variables and try a solution of the form \( V^c(x, q, R) = f(x)g(q)h(R) \), which leads to:

\[
e^{2\beta x} \left[ \frac{1}{2} f''(x) - \frac{1}{2} f'(x) + \frac{g'(q)}{g(q)} \right] = \frac{h'(R)}{h(R)}, \tag{4.8}
\]

assuming \( \frac{h'(R)}{h(R)} = c_R \), gives us \( h(R) = e^{c_R R} \) where \( c_R \) is an arbitrary constant to be determined. We use the subscript \( R \) just to show which part of the separation of variables it arises from. If we let \( \frac{g'(q)}{g(q)} = c_q \), then (4.8) becomes:

\[
f''(x) - f'(x) + 2f(x)c_q - 2c_R e^{-2\beta x} f(x) = 0. \tag{4.9}
\]

Let \( f(x) = e^{x/2} \hat{f}(x) \),

\[
\frac{df(x)}{dx} = \frac{1}{2} e^{x/2} \hat{f}(x) + e^{x/2} \hat{f}'(x)
\]

\[
\frac{d^2 f(x)}{dx^2} = \frac{1}{4} e^{x/2} \hat{f}(x) + \frac{1}{2} e^{x/2} \hat{f}'(x) + \frac{1}{2} e^{x/2} \hat{f}''(x) + e^{x/2} \hat{f}''(x)
\]

\[
= \frac{1}{4} e^{x/2} \hat{f}(x) + e^{x/2} \hat{f}'(x) + e^{x/2} \hat{f}''(x). \tag{4.10}
\]

Equation (4.9) becomes:

\[
\frac{1}{4} \hat{f}(x) + \hat{f}'(x) + \hat{f}''(x) - \frac{1}{2} \hat{f}(x) - \hat{f}'(x) + 2f(x)c_q - 2c_R e^{-2\beta x} \hat{f}(x) = 0
\]

\[
\hat{f}''(x) - \hat{f}(x) \left[ \frac{1}{4} - 2c_R + 2e^{-2\beta x} c_R \right] = 0. \tag{4.11}
\]
Let \( \hat{x} = -\sqrt{2cRe^{-\beta x}} \)

\[
\begin{align*}
\frac{d\hat{x}}{dx} &= \sqrt{2cRe^{-\beta x}} \left( \frac{d\hat{x}}{dx} \right)^2 = 2cRe^{-2\beta x} = \beta^2 \hat{x}^2 \\
\frac{d^2\hat{x}}{dx^2} &= -\beta \sqrt{2cRe^{-\beta x}} = \beta^2. \tag{4.12}
\end{align*}
\]

We also introduce, with the motivation to obtain a Bessel-type equation, \( \hat{f}(\hat{x}) = \hat{f}(x) \),

whose derivatives evaluate to

\[
\begin{align*}
\frac{d\hat{f}(x)}{dx} &= \frac{d\hat{f}(\hat{x})}{d\hat{x}} \frac{d\hat{x}}{dx} \\
\frac{d^2\hat{f}(x)}{dx^2} &= \frac{d^2\hat{f}(\hat{x})}{d\hat{x}^2} \left( \frac{d\hat{x}}{dx} \right)^2 + \frac{d\hat{f}(\hat{x})}{d\hat{x}} \frac{d^2\hat{x}}{dx^2} \\
&= \hat{f}''(\hat{x}) \beta^2 \hat{x}^2 + \hat{f}'(\hat{x}) \beta^2 \hat{x}. \tag{4.13}
\end{align*}
\]

This changes (4.11) to

\[
\hat{f}''(\hat{x}) \beta^2 \hat{x}^2 + \hat{f}'(\hat{x}) \beta^2 \hat{x} - \hat{f}(\hat{x}) \left[ \frac{1}{4} - 2c_q + \beta^2 \hat{x}^2 \right] = 0. \tag{4.14}
\]

or

\[
\hat{x}^2 \hat{f}''(\hat{x}) + \hat{x} \hat{f}'(\hat{x}) - \hat{f}(\hat{x}) \left[ \frac{1}{4} - 8c_q \beta^2 \right] = 0. \tag{4.15}
\]

We can compare this with the modified Bessel’s equation:

\[
x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - (x^2 + \alpha^2)y = 0, \tag{4.16}
\]

which has solutions \( I_{\alpha}(x) \) and \( K_{\alpha}(x) \). We have

\[
\alpha^2 = \frac{1}{4} - 8c_q \beta^2, \quad \beta \neq 0 \tag{4.17}
\]

and general solutions of the form:

\[
\hat{f}(\hat{x}) = a_1 I_{\alpha}(\hat{x}) + a_2 K_{\alpha}(\hat{x}). \tag{4.18}
\]

Putting this together with our \( g(q) \) and \( h(R) \) solutions, we have general solutions of PDE (4.8) of the form:

\[
V^e(x, q, R) = a_1 e^{x/2} I_{\alpha} \left( \frac{-\sqrt{2cRe^{-\beta x}}}{\beta} \right) e^{\eta q} e^{ER^2} + a_2 e^{x/2} K_{\alpha} \left( \frac{-\sqrt{2cRe^{-\beta x}}}{\beta} \right) e^{\eta q} e^{ER^2}. \tag{4.19}
\]
Finally, putting in the definitions of \( x = \log \left( \frac{F_t}{F_0} \right) \),

\[
q = \int_0^t d \langle Y \rangle_s = \int_0^t s^2 \left( \log \frac{F_s}{F_0} \right) d\tau_s = \int_0^t \left( \frac{F_s}{F_0} \right)^{2\beta} \alpha_s^2 ds,
\]

and \( R = H - \tau_t = H - \int_0^t \alpha_s^2 ds \), the general solution (6.2) becomes:

\[
V^c(F_t, t) = a_1 \left( \frac{F_t}{F_0} \right)^{1/2} I_\alpha \left( -\frac{\sqrt{2c_R}}{\beta} \left( \frac{F_t}{F_0} \right)^{-\beta} \right) e^{q_0 t} \left( \frac{F_t}{F_0} \right)^{2\beta} \alpha_s^2 ds e^{c_R \left( H - \int_0^t \alpha_s^2 ds \right)}
\]

\[
+ a_2 \left( \frac{F_t}{F_0} \right)^{1/2} K_\alpha \left( -\frac{\sqrt{2c_R}}{\beta} \left( \frac{F_t}{F_0} \right)^{-\beta} \right) e^{q_0 t} \left( \frac{F_t}{F_0} \right)^{2\beta} \alpha_s^2 ds e^{c_R \left( H - \int_0^t \alpha_s^2 ds \right)}.
\]

We now turn our attention to investigating the behaviour of this solution in the \( x = \log \left( \frac{F_t}{F_0} \right) \) variable, to see if we can rule out one of the kinds of modified Bessel functions \( I_\alpha(x) \) and \( K_\alpha(x) \). In appendix (A.3.2) we plot below the behaviour of \( e^{x/2}K_\alpha(e^{x/2}) \) and \( e^{x/2}I_\alpha(e^{x/2}) \) for various values of \( \alpha \) and \( \beta = -1/2 \). If we impose a payoff condition \( h(q) = \min(q, k) \) then we should require that the solution does not grow too quickly for large values of \( x \), so we can rule out the \( I_\alpha(x) \) set of solutions, leaving us with a general solution of the form:

\[
V^c(x, q, R) = \int_{R^+} A_x c_{q,R} e^{x/2}K_\alpha \left( -\frac{\sqrt{2c_R}e^{-\beta x}}{\beta} \right) e^{c_R \left( H - \int_0^t \alpha_s^2 ds \right)} \]

To satisfy the pay-off condition at expiry \( \tau = H \), i.e. \( R = 0 \), we need to find coefficients \( A_{x,q,R} \) which satisfy:

\[
h(q) = \int_{R^+} A_{x,q,R} e^{x/2}K_\alpha \left( -\frac{\sqrt{2c_R}e^{-\beta x}}{\beta} \right) e^{c_R \left( H - \int_0^t \alpha_s^2 ds \right)} \]

We use the following integral from [47, 1] to help us perform the \( c_R \) integral:

\[
\int_0^\infty y^\mu K_\nu(y)dy = 2^{\mu-1}a^{\mu-1}\Gamma \left( \frac{1+\mu+\nu}{2} \right) \Gamma \left( \frac{1+\mu-\nu}{2} \right). \]

(4.24)

For concreteness, we take \( \beta = -1/2 \). Let \( y = \sqrt{c_R} \), \( dy = \frac{1}{2\sqrt{c_R}}dc_R \) i.e. \( dc_R = 2ydy \). We assume \( A_{x,q,R} = A_{x}A_{c,R} \) and \( A_{c,R} = cy^\mu \) and we let \( a = 2\sqrt{2}e^{x/2} \). Our goal is that the \( c_R \) integral will remove the \( x \) dependence, and leave us with a simple integral in \( c_q \) to match with the terminal condition,

\[
\int_0^\infty 2cy^{\mu+1} K_\alpha(ay)dy = 2c^{2}\frac{a}{2\sqrt{2}} \Gamma \left( \frac{2+\mu+\alpha}{2} \right) \Gamma \left( \frac{2+\mu-\alpha}{2} \right).
\]

(4.25)
For independence of $x$ (i.e. independence of $a$), we take $\mu = -1$. The left hand side then simplifies to
\[ \frac{c}{2\sqrt{2}} \Gamma \left(\frac{1 + \alpha}{2}\right) \Gamma \left(\frac{1 - \alpha}{2}\right). \] (4.26)
If we now take $c = \frac{2\sqrt{2}}{\Gamma \left(\frac{1 + \alpha}{2}\right) \Gamma \left(\frac{1 - \alpha}{2}\right)}$, we can simplify the terminal condition as follows:
\[ h(q) = \int_{\mathbb{R}^+} \int_{\mathbb{R}} A_{c_q} \frac{2\sqrt{2}}{\Gamma \left(\frac{1 + \alpha}{2}\right) \Gamma \left(\frac{1 - \alpha}{2}\right)} e^{s/2} K_{\alpha} \left(2\sqrt{2}e^{s/2}y\right) e^{c_q q} dy dc_q = \int_{\mathbb{R}} A_{c_q} e^{c_q q} dc_q. \] (4.27)

In principle, we could then find coefficients $A_{c_q}$ such that the integral evaluates to the terminal condition. We would then have a semi-analytical solution as a double integral, which could be at least be evaluated numerically. We don’t pursue this further in this work, but instead look at solving the original PDE (6.1) using an explicit finite difference scheme, which can in principle be done for any suitable choice of skew function $s(x)$.

### 4.2 Martingale Representations of the clock

We have found already that we can replicate the pay-off of a claim which pays out $h(Y_T, \langle Y \rangle_T)$ at expiry without the need to buy or sell options on the futures. This required knowledge of the value of the stochastic clock at expiry ex-ante to be some positive constant which we called $H$. This was rather unrealistic since in general, the risk-neutral probability distribution function (PDF) of the stochastic clock is unknown at time zero. Carr and Lee show that we can infer this PDF which we denote as $q_t = \mathbb{Q}\{\tau_t \in dH|\mathcal{F}_t\}$ from an observation at $t \in [0, T]$ of the market prices of options on the futures price maturing at $T$. We have $Y_t = \log(F_t/F_0)$, and from (3.1) we have
\[ dY_t = -\frac{s^2(Y_t)}{2} dt + s(Y_t) dW_t, \quad t \in [0, T]. \] (4.28)

We can now see $Y_t$ is a process evolving with the stochastic clock: $Y_t = X_{\tau_t}$, $t \in [0, T]$, where $X$ solves the SDE:
\[ dX_{\tau_t} = -\frac{s^2(X_{\tau_t})}{2} d\tau_t + s(X_{\tau_t}) dW_{\tau_t}, \quad \tau_t \geq 0 \text{ subject to } X_0 = 0. \] (4.29)
Taking the quadratic variation, we find $d\langle X \rangle_{\tau_t} = s^2(X_{\tau_t}) d\tau_t$. The next step is to find the infinitesimal generator of $X$ which we denote $\mathcal{G}$, whose eigenfunctions $\phi(x)$ will define our pay-off, hence Carr and Lee’s term “eigenclaim”. The infinitesimal generator of $X$ is:
\[ \mathcal{G}\phi(x) = \frac{s^2 d^2\phi(x)}{2 dx^2} - \frac{s^2(x)}{2} \frac{d\phi(x)}{dx} = \lambda \phi(x), \] (4.30)
where $\lambda$ are the eigenvalues, and we can think of a payoff $\phi(x, \lambda)$. To illustrate this and continuing our CEV example, we can find concrete eigenvalues and eigenfunctions of this equation. In the CEV case we have $s(x) = \exp(\beta x)$

$$\frac{e^{2\beta x}}{2} \frac{d^2 \phi(x)}{dx^2} - \frac{e^{2\beta x}}{2} \frac{d\phi(x)}{dx} - \lambda \phi(x) = 0. \quad (4.31)$$

Let

$$\Psi(x) = e^{-x/2} \phi(x)$$

$$\Psi'(x) = -\frac{1}{2} e^{-x/2} \phi(x) + e^{-x/2} \phi'(x)$$

$$\Psi''(x) = \frac{1}{4} e^{-x/2} \phi(x) - \frac{1}{2} e^{-x/2} \phi'(x) - \frac{1}{2} e^{-x/2} \phi'(x) + e^{-x/2} \phi''(x)$$

$$= \frac{1}{4} e^{-x/2} \phi(x) - e^{-x/2} \phi'(x) + e^{-x/2} \phi''(x). \quad (4.32)$$

Putting these into 4.31 we have

$$e^{2\beta x} \left( \frac{1}{4} e^{x/2} \Psi + e^{x/2} \Psi' + e^{x/2} \Psi'' \right) - e^{2\beta x} \left( \frac{1}{2} e^{x/2} \Psi + e^{x/2} \Psi' \right) - \lambda e^{x/2} \Psi = 0, \quad (4.33)$$

which simplifies to

$$\Psi'' - \Psi \left[ 2\lambda e^{-2\beta x} + \frac{1}{4} \right] = 0. \quad (4.34)$$

If we now define $\hat{x} = -\frac{\sqrt{2\sqrt{\lambda}e^{-\beta x}}}{\beta}$, we have $\hat{x}^2 = \frac{2\lambda e^{-2\beta x}}{\beta^2}$ and

$$\frac{d\hat{x}}{dx} = \sqrt{2\sqrt{\lambda}e^{-\beta x}} = -\beta \hat{x}$$

$$\frac{d^2 \hat{x}}{dx^2} = -\beta \sqrt{2\sqrt{\lambda}e^{-\beta x}} = \beta^2 \hat{x}. \quad (4.35)$$

We now define $\Psi(x) = \hat{\Psi} (\hat{x})$

$$\frac{d\Psi(x)}{dx} = \frac{d\hat{\Psi}(\hat{x})}{d\hat{x}} \frac{d\hat{x}}{dx} = -\frac{d\hat{\Psi}(\hat{x})}{d\hat{x}} \beta \hat{x}$$

$$\frac{d^2 \Psi(x)}{dx^2} = \frac{d^2 \hat{\Psi}(\hat{x})}{d\hat{x}^2} \left( \frac{d\hat{x}}{dx} \right)^2 + \frac{d\hat{\Psi}(\hat{x})}{d\hat{x}} \frac{d^2 \hat{x}}{dx^2}$$

$$= \frac{d^2 \hat{\Psi}(\hat{x})}{d\hat{x}^2} \hat{x}^2 \beta^2 + \frac{d\hat{\Psi}(\hat{x})}{d\hat{x}} \beta^2 \hat{x}. \quad (4.36)$$

Inserting these into (4.34) we have

$$\hat{\Psi}'' \hat{x}^2 + \hat{\Psi}' \hat{x} - \hat{\Psi} \left[ \hat{x}^2 + \frac{1}{4\beta^2} \right] = 0. \quad (4.37)$$
Comparing this with the modified Bessel equation
\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \alpha^2) = 0 \]  
shows us that we have solutions of the form
\[ \hat{\Psi}(\hat{x}) = a_1 I_\alpha(\hat{x}) + a_2 K_\alpha(\hat{x}), \]  
with \( \alpha = \frac{1}{2\beta} \) and \( \hat{x} = -\frac{\sqrt{2}\sqrt{\lambda} e^{-\beta x}}{\beta} \) rewinding the steps we find eigenfunctions \( \phi(x) \) of the form
\[ \phi(x) = e^{x/2} \hat{\Psi}(\hat{x}) = e^{x/2} \left[ a_1 I_{\frac{1}{2\beta}} \left( -\frac{\sqrt{2}\sqrt{\lambda} e^{-\beta x}}{\beta} \right) \right] + \ldots \]  
and inserting the definition of \( x \) gives us solutions of the form
\[ \phi(F_t; \lambda) = \left( \frac{F_t}{F_0} \right)^2 \left[ a_1 I_{\frac{1}{2\beta}} \left( -\frac{\sqrt{2}\sqrt{\lambda} e^{-\beta x}}{\beta} \right) \right] + \ldots. \]  

Without the restriction to the CEV example just given, the payoff can be replicated statically by trading in options of all strikes. This follows a similar argument to that of section (2.6.2), and can be found in the appendix of Carr and Lee, but we reproduce it here for clarity. Let
\[ f(F; \lambda) \equiv \phi \left( \log \frac{F}{F_0}; \lambda \right), \quad F > 0, \]  
giving us
\[ f'(F; \lambda) = \phi' \left( \log \frac{F}{F_0}; \lambda \right) \frac{1}{F} \]  
and
\[ f''(F_t; \lambda) = \left[ \phi'' \left( \log \frac{F}{F_0}; \lambda \right) - \phi' \left( \log \frac{F}{F_0}; \lambda \right) \right] \frac{1}{F^2}. \]  

Using the same techniques as in section (2.6.2), we can expand \( f \) in \( F \) about \( F_0 \):
\[ f(F; \lambda) = f(F_0; \lambda) + f'(F_0; \lambda)(F - F_0) + \int_0^{F_0} f''(K; \lambda)(K - F)^+ dK + \int_{F_0}^{\infty} f''(K; \lambda)(F - K)^+ dK. \]  
Hence, from (4.42) we have:
\[ \hat{\phi}(Y; \lambda) = \hat{\phi}(0; \lambda) + \frac{\phi'(0; \lambda)}{F_0} [(F - F_0)^+ - (F_0 - F)^+] + \int_0^{F_0} \frac{\phi''(\log \frac{K}{F_0}; \lambda)}{K^2} (K - F)^+ dK + \int_{F_0}^{\infty} \frac{\phi''(\log \frac{K}{F_0}; \lambda)}{K^2} (F - K)^+ dK. \]  

24
This equation shows that we can replicate the payoff by holding risk-free assets, \( \frac{\phi'(0; \lambda)}{F_0} \) calls with strike \( F_0 \) (initially at the money), and \( \frac{\phi''(\log \frac{K}{F_0}; \lambda)}{K^2} \) puts with strike \( F_0 \) (initially at the money), \( \frac{\phi''(\log \frac{K}{F_0}; \lambda)}{K^2} (K - F)^+ dK \) puts for each strike \( K \in (0, F_0) \) (initially out of the money) and \( \frac{\phi''(\log \frac{K}{F_0}; \lambda)}{K^2} (F - K)^+ dK \) calls for each strike \( K \in (F_0, \infty) \) (initially out of the money). For times \( t \in [0, T] \), no arbitrage means we have

\[
V_t^\phi(\lambda) \equiv E[\phi(Y_T; \lambda)|\mathcal{F}_t] = \phi(0; \lambda) + \frac{\phi'(0; \lambda)}{F_0} [C_t(F_0, T) - P_t(F_0, T)] + \\
\int_{F_0}^{0} \frac{\phi''(\log \frac{K}{F_0}; \lambda) - \phi'(\log \frac{K}{F_0}; \lambda)}{K^2} P_t(K, T) dK + \int_{F_0}^{\infty} \frac{\phi''(\log \frac{K}{F_0}; \lambda) - \phi'(\log \frac{K}{F_0}; \lambda)}{K^2} C_t(K, T) dK
\]

(4.47)

where \( P_t(K, T) \) and \( C_t(K, T) \) denote the market prices at time \( t \in [0, T] \) of European puts and calls of strike \( K > 0 \) and maturity \( T \geq t \).

Carr and Lee use the Bromwich inversion formula [1]:

\[
h(\tau T) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\tau \tau T} \hat{h}(\lambda) d\lambda,
\]

(4.48)

together with the Laplace transform of the delta function [1] \( \delta(\tau T - H) \)

\[
\mathcal{L}(\delta(\tau - H))(\lambda) = \int_{0}^{\infty} e^{-\lambda \tau} \delta(\tau - H) d\tau = e^{-\lambda H}, \lambda \in \mathbb{C},
\]

(4.49)

and take expectations under the risk neutral measure to obtain

\[
\mathbb{E}[\delta(\tau T - H)|\mathcal{F}_t] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbb{E}[Re[e^{-(a+iw)H} e^{(a+iw)\tau T}] dw|\mathcal{F}_t]
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} Re[e^{-(a+iw)H} \mathbb{E}[e^{(a+iw)\tau T}] |\mathcal{F}_t] dw
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} Re \left[ e^{(a+iw)(\tau T - H)} V_t^\phi(a + iw) \frac{\phi(Y_t; a + iw)}{\phi(Y_t; a + iw)} \right] dw.
\]

(4.50)

To bring this section back to finance, we can see that substituting (4.47) into the last equation will give us the conditional risk neutral density of the terminal clock in terms of option prices expiring at that time. Moreover, initially, we can read the initial risk neutral probability density distribution of \( \tau_T \) from the initial implied volatility smile – since we are integrating over the full set of strikes which will include prices reflecting the skew.

\[\text{Carr and Lee refer to this as an Eigenpayoff}^2\]
4.3 Pricing Quadratic variation derivatives – stochastic clock terminal value unknown, using stochastic clock probability density function

We have discussed how we could price derivatives which paid a function of the quadratic variation of a process which was running on a stochastic clock whose terminal value was known at the start (6.2). In the more realistic case we don’t know if the stochastic clock is going to reach a particular value and we would need to know the risk-neutral probability density function (pdf) for the clock which we could integrate our solution (6.2) against to obtain a price of an option on quadratic variation. We discussed in the last section how the clock’s pdf could be inferred from contemporaneous option prices. In fact, we showed (repeating the steps of Carr and Lee) that at time 0, the initial risk-neutral pdf of \( \tau_T \) could be read off from the implied volatility smile. The stochastic clock introduces stochastic volatility into the model, and the fact we new need the set of options in a similar way to the work of Derman et al. illustrates we are using the full shape of the implied volatility surface in our model, but we have the added feature that Carr and Lee’s framework holds for many possible choices of the skew function – i.e. many possible SDEs for the process, not just Black-Scholes. Following again the notation of Carr and Lee, let \( V_t^h \) denote the value at time \( t \in [0, T] \) of a derivative paying \( h(Y_T, \langle Y \rangle_T) \) at expiry \( T \). Taking the risk-neutral expectation of this pay-off under the filtration \( \mathcal{F}_t \) we have

\[
V_t^h = \mathbb{E}^Q[h(Y_T, \langle Y \rangle_T)|\mathcal{F}_t].
\]

Defining

\[
p_t(H) = \mathbb{Q}\{\tau_T \in dH|\mathcal{F}_t}\]

as the risk-neutral pdf of the terminal stochastic clock as the price at time \( t \in [0, T] \) of a butterfly spread (pay-off\(^3\) as in (4.50)) on the stochastic clock at expiry with strike \( H \geq 0 \). Carr and Lee’s assumption that \( \alpha \) defined in (3.2) evolves independently of \( W \), changes in the futures price and changes in the prices of these butterfly spreads are all conditionally independent

\[
\langle F, p(H) \rangle_t = 0.
\]

With knowledge of the risk-neutral probability density function of the stochastic clock \( p_t(H) \), together with solutions \( V^\alpha(Y_t, \langle Y \rangle_t, H - \tau_t) \) of the PDE (6.1) which are for the

\(^3\)E.g. Long one call with a strike price of \( (H - \delta H) \), short two calls with a strike price of \( H \), long one call with a strike price of \( (H + \delta H) \)
case in which we know the initial and terminal values of the stochastic clock, we can as a guess take it that smearing the $V^c(Y_t, \langle Y \rangle_t, H - \tau_t)$ with the $p_t(H)$ and integrating over all possible $H$ should give us prices for options which are contingent on the stochastic clock any particular time barrier. That is

$$V^h_t = \int_0^\infty V^c(Y_t, \langle Y \rangle_t, H - \tau_t) p_t(H) dH.$$  

Now differentiating both sides

$$dV^h_t = \int_0^\infty dp_t(H) V^c(Y_t, \langle Y \rangle_t, H - \tau_t) dH + \int_0^\infty p_t(H) dV^c(Y_t, \langle Y \rangle_t, H - \tau_t) dH$$

$$+ \int_0^\infty dp_t(H) dV^c(Y_t, \langle Y \rangle_t, H - \tau_t) dH.$$  \hspace{1cm} (4.51)

Carr and Lee discuss how well-defined this expression is, so we won’t ponder on this here. The fact that $V^c(x, q, R)$ solves (6.1) means that

$$dV^c(Y_t, \langle Y \rangle_t, H - \tau_t) = \frac{\partial}{\partial x} V^c(Y_t, \langle Y \rangle_t, H - \tau_t) \frac{dF_t}{F_t}, \quad t \in [0, T]$$

since all other terms vanish. We now substitute (4.3) and (4.3) into (4.51) to obtain

$$dV^h_t = \int_0^\infty dp_t(H) V^c(Y_t, \langle Y \rangle_t, H - \tau_t) dH + \int_0^\infty p_t(H) \frac{\partial}{\partial x} V^c(Y_t, \langle Y \rangle_t, H - \tau_t) \frac{dF_t}{F_t} dH$$

and finally integrating gives us

$$h(Y_T, \langle Y \rangle_T) = V^h_0 + \int_0^T \int_0^\infty dp_t(H) V^c(Y_t, \langle Y \rangle_t, H - \tau_t) dH dt$$

$$+ \int_0^\infty \int_0^T p_t(H) \frac{\partial}{\partial x} V^c(Y_t, \langle Y \rangle_t, H - \tau_t) dH \frac{1}{F_t} dF_t.$$  \hspace{1cm} (4.53)

where

$$V^h_0 \equiv \int_0^\infty p_0(H) V^c(0, 0, H) dH.$$  

This shows that we can replicate a payoff on the terminal log price and the terminal quadratic variation by charging an initial amount $V^h_0$ and by dynamically trading in $\int_0^\infty p_t(H) \frac{\partial}{\partial x} V^c(Y_t, \langle Y \rangle_t, H - \tau_t) dH \frac{1}{F_t}$ futures and $V^c(Y_t, \langle Y \rangle_t, H - \tau_t) dH$ butterfly spreads on the terminal clock of all strikes $H \geq \tau_t$. The value of the clock $\tau_t$ is known from (3.7) i.e. from the path taken. Equation (6.5) brings together much of the work in this thesis: The initial price is in this equation, the initial probability density $p_0(H)$ can be obtained by setting $t = 0$ in (4.50) and we can use the solutions obtained
for (6.1) to obtain the $V^c(x, q, R)$ entering the equation. We found in (6.2) general solutions in the CEV case, but without solving for particular boundary conditions – we just postulated that it should in principle be possible for simpler cases like CEV. We will show in the next chapter that we can also obtain the important $V^c(x, q, R)$ using numerical methods. We will do this for the CEV case too, with specific realistic boundary conditions.
Chapter 5
Numerical Methods

PDE equations can only be solved in a limited number of cases, and often in the physical sciences and finance, numerical methods are employed to tackle the problem directly. We first summarise the techniques we will use to find a numerical solution, and use these techniques on PDE (6.1) to obtain numerical solutions.

5.1 Finite difference

One of the most common methods to solve PDEs is with Finite Difference methods [69, 65, 66, 64, 17]. The idea is to discretise both the solution, \( V \), and the equation itself, i.e. to approximate the solution by a finite set of values and to replace the continuous PDE with a linear system of equations for these values. In the Black-Scholes world, our solution would be a function of the underlying value \( S \) and time \( t \). Our grid would be made up of \( S_0 = 0, S_1 = \Delta S, S_2 = 2\Delta S, \ldots, S_N = N\Delta S \), where \( \Delta S = 1/N \), and for the time-steps we have \( t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \ldots, t_M = M\Delta t \), where \( \Delta t = 1/M \). The numerical approximation for the solution is then \( V_{n,m} = V(S_n, t_m) = V(n\Delta S, m\Delta t) \). We would then have an appropriate boundary condition representing the pay-off of the option e.g. \( V_{n,M} = max(K - S_n, 0) = max(K - n\Delta S, 0) \), and some financially sensible other conditions which will help to keep the solutions unique and hence the implementation more stable e.g. for a put-option, we would expect that when the underlying is very large, the change of getting a positive payout will be small, so we can incorporate this by saying \( V_{N,M} = 0 \). The next step is to approximate the derivatives in our PDE by finite differences at grid points. Common choices are [69]: right-sided difference

\[
\frac{\partial V}{\partial S}(S_n, t_m) = \frac{V(S_{n+1}, t_m) - V(S_n, t_m)}{\Delta S} + O(\Delta S)
\]
left-sided difference
\[ \frac{\partial V}{\partial S}(S_n, t_m) = \frac{V(S_n, t_m) - V(S_{n-1}, t_m)}{\Delta S} + O(\Delta S) \]

central difference
\[ \frac{\partial V}{\partial S}(S_n, t_m) = \frac{V(S_{n+1}, t_m) - V(S_{n-1}, t_m)}{2\Delta S} + O(\Delta S^2) . \]

Extending this to the second derivative, we have
\[ \frac{\partial^2 V}{\partial S^2}(S_n, t_m) = \frac{V(S_{n+1}, t_m) - 2V(S_n, t_m) + V(S_{n-1}, t_m)}{\Delta S^2} + O(\Delta S^2) . \]

For the time derivatives, common choices are: forward difference
\[ \frac{\partial V}{\partial t}(S_n, m\Delta t) = \frac{V(S_n, (m+1)\Delta t) - V(S_n, m\Delta t)}{\Delta t} + O(\Delta t) \]

backward difference
\[ \frac{\partial V}{\partial t}(S_n, m\Delta t) = \frac{V(S_n, m\Delta t) - V(S_n, (m-1)\Delta t)}{\Delta t} + O(\Delta t) \]

central difference
\[ \frac{\partial V}{\partial t}(S_n, m\Delta t) = \frac{V(S_n, (m+1)\Delta t) - V(S_n, (m-1)\Delta t)}{2\Delta t} + O(\Delta t^2) \]

When we use the central differences for the derivatives involving the underlying, the above three time derivative definitions give rise to the three well known schemes: explicit Euler, implicit Euler and leapfrog scheme. For example, for the Black-Scholes PDE, the explicit Euler scheme would be
\[ \frac{V^{m-1} - V^m}{\Delta t} - \frac{1}{2} \sigma^2 \Delta S^2 n^2 \frac{V^m_{n+1} - 2V^m_n + V^m_{n-1}}{\Delta S^2} - r \Delta S n \frac{V^m_{n+1} - V^m_{n-1}}{2\Delta S} + r V^m_n = 0 . \]

with \( m = 1, \ldots, M \). For the implicit scheme, we would have
\[ \frac{V^m - V^{m+1}}{\Delta t} - \frac{1}{2} \sigma^2 \Delta S^2 n^2 \frac{V^m_{n+1} - 2V^m_n + V^m_{n-1}}{\Delta S^2} - r \Delta S n \frac{V^m_{n+1} - V^m_{n-1}}{2\Delta S} + r V^m_n = 0 \]

with \( m = 0, \ldots, M - 1 \), and for the leapfrog scheme, we would have
\[ \frac{V^{m-1} - V^{m+1}}{2\Delta t} - \frac{1}{2} \sigma^2 \Delta S^2 n^2 \frac{V^{m+1}_{n+1} - 2V^m_n + V^{m-1}_{n-1}}{\Delta S^2} - r \Delta S n \frac{V^{m+1}_{n+1} - V^{m-1}_{n-1}}{2\Delta S} + r V^m_n = 0 \]

with \( m = 1, \ldots, M - 1 \)

For the explicit scheme, we can compute \( V^{m-1} \) from the values of \( V^{m-1}, V^m, V^{m+1}_n \) as
\[ V^{m-1} = A^m_n V^{m+1}_{n-1} + B^m_n V^m_n + C^m_n V^{m+1}_n \]
for \( n = 1, \ldots, N \), with

\[
A^n = \frac{1}{2} n^2 \sigma^2 \Delta t - \frac{1}{2} nr \Delta t
\]

\[
B^n = 1 - n^2 \sigma^2 \Delta t - r \Delta t
\]

\[
C^n = \frac{1}{2} n^2 \sigma^2 \Delta t + \frac{1}{2} nr \Delta t
\]

(5.1)

Often is necessary to find cases where our model leads to a simpler solution, perhaps at a boundary, which we can then use to stabilise our numerical solution, capture better the real world behaviour and make the numerical solution unique. Continuing the Black-Scholes example, when \( n = 0 \) above, this is the same as having an underlying price of zero, which acts as a simplifier to the above equations. We reduce the above to

\[
V_0^{m-1} = (1 - r \Delta t)V_0^m
\]

For put options i.e with pay-off \( V^M_n = \max(K - S_n, 0) = \max(K - n \Delta S, 0) \), we have a physical boundary condition that \( V_N^{m-1} = 0 \) since nobody would buy a put option which is never going to expire in the money. We use the pay-off condition, and work inductively in \( m \) to compute the full solution down to time \( t = 0 \).

### 5.2 Application to options on quadratic variation with skew

The above explicit scheme can be used to solve (6.1) as a complement to the analytical solution (6.2) we obtained earlier. We will outline how this is done, the assumptions we make, and include the matlab code implementation in the appendix – also see attached Matlab file ExplicitEulerQVD3March12.m. To recap, in (6.1), we have a derivative whose pay-off depends on the quadratic variation of the underlying process. Expiry is reached when the stochastic clock which appears directly in the process SDE, reaches a barrier \( H \), if the clock never reaches \( H \), the derivative pays nothing. To remind the reader, the PDE in the CEV case which we want to solve is

\[
e^{2 \beta x} \left[ \frac{1}{2} \frac{\partial^2 V^c(x, q, R)}{\partial x^2} - \frac{1}{2} \frac{\partial V^c(x, q, R)}{\partial x} + \frac{\partial V^c(x, q, R)}{\partial q} \right] = \frac{\partial V^c(x, q, R)}{\partial R}
\]

(5.2)

with initial condition \( V^c(x, q, 0) = h(q) \), where \( h(q) \) is the pay-off. We will at numerical solutions in two cases we mentioned before, which are typical pay-offs:

- volatility swap \( h(x, q) = \sqrt{q} \)
capped variance swap $h(x, q) = \min(q, k)$ where $k > 0$ is the positive cap.

We rewrite the PDE with $\tau$ instead of $R$, discretise our coordinates $x = n\Delta x, q = l\Delta q, \tau = m\Delta \tau$ and use the explicit finite difference scheme. We need a small enough time step in comparison to the $x$-mesh – which can be derived from the standard stability condition for explicit schemes (see below). The $q$ direction is purely hyperbolic, so we will need a one-sided difference e.g. upwind difference$^1$. Since we have one-way flow in the $q$-direction, we will need an asymptotic boundary condition for large $q$ but not at $q=0$. The PDE becomes

$$e^{2\beta n\Delta x} \left( \frac{1}{2} \frac{V_{n+1,l,m} - 2V_{n,l,m} + V_{n-1,l,m}}{\Delta x^2} - \frac{1}{2} \frac{V_{n+1,l,m} - V_{n-1,l,m}}{2\Delta x} + \frac{V_{n,l,m} - V_{n,l-1,m}}{\Delta q} \right) = -\left( \frac{V_{n,l,m} - V_{n,l,m-1}}{\Delta \tau} \right).$$  \hspace{1cm} (5.3)

Expressing how we obtain the previous time step $m-1$ term from time step terms $m$, we have

$$V_{n,l,m-1} = A_{n,l,m}V_{n+1,l,m} + B_{n,l,m}V_{n,l,m} + C_{n,l,m}V_{n-1,l,m} + D_{n,l,m}V_{n,l-1,m} \hspace{1cm} (5.4)$$

where

$$A_{n,l,m} = \left( \frac{1}{2} \frac{e^{2\beta n\Delta x} \Delta \tau}{(\Delta x)^2} - \frac{e^{2\beta n\Delta x} \Delta \tau}{4\Delta x} \right),$$

$$B_{n,l,m} = \left( 1 - \frac{e^{2\beta n\Delta x} \Delta \tau}{(\Delta x)^2} + \frac{e^{2\beta n\Delta x} \Delta \tau}{4\Delta q} \right),$$

$$C_{n,l,m} = \left( \frac{1}{2} \frac{e^{2\beta n\Delta x} \Delta \tau}{(\Delta x)^2} + \frac{e^{2\beta n\Delta x} \Delta \tau}{4\Delta x} \right),$$

$$D_{n,l,m} = -\left( \frac{e^{2\beta n\Delta x} \Delta \tau}{\Delta q} \right).$$  \hspace{1cm} (5.5)

### 5.3 Stability Analysis

For stable solutions in the explicit finite difference framework, one has to play carefully with the discretisation in the space and time directions. We observe that we have

$$A_{n,l,m} + B_{n,l,m} + C_{n,l,m} + D_{n,l,m} = 1$$

and we would have that the LHS is less than 1 if we had introduced non-zero interest rates. If we can achieve that

$$A_{n,l,m}, B_{n,l,m}, C_{n,l,m} > 0$$

$^1$Thanks to Dr. Christoph Reisinger for suggesting this to me.
then $V_{n,l,m-1} \geq 0$ and $V_{n,l,m-1} \leq \max(V_{n-1,l,m}, V_{n+1,l,m}, V_{n1,l,m}, V_{n-1,l,m})$. For this to be true, we require

$$1 \geq \Delta \tau \left( \frac{e^{2\beta n \Delta x}}{\Delta x^2} - \frac{2e^{2\beta n \Delta x}}{\Delta q} \right)$$

where we have used $D_{n,l,m} = -\left( \frac{e^{2\beta n \Delta x} \Delta \tau}{\Delta q} \right) \leq 0$. This leads to

$$\Delta \tau \leq \frac{e^{-2\beta n \Delta x} \Delta x^2 \Delta q}{\Delta q - \Delta x^2}.$$ 

Taking some typical values for $X$, constrains us in our choice of $\Delta q$, $\Delta x$ and $\Delta \tau$. We choose appropriate values in our numerical simulations which can be found in appendix (A.4). These restrictions have to also be within the range of the computational power and memory to hold the solution matrices in memory. Our errors in $X$ are $O(1/N^2)$, in $Q$ are $O(1/L)$ and in $\tau$ are $O(1/M)$. With $N = L$ and $M \approx N^2$, the largest error is $O(1/L)$.

### 5.4 Numerical solutions of PDE (6.1)

We plot some figures below showing solutions at various time points, $\beta$ parameters and for pay-offs $\min(q,k)$ and $\sqrt{q}$:

![Figure 5.1: Solution of PDE (6.1) at $\tau = 0$ with pay-off $\min(q,10)$, $\beta = -1/2$.](image)

Note, we don’t have any discounting in this framework – we have set the risk-free rate to zero, hence no decaying is observed in the plots. In this framework, we know the start and end values of the stochastic clock to be $0$ and $H$ respectively – we observe that for pay-offs solely dependent on the quadratic variation, there is no change in the price over various times. We also observe that as time gets closer to expiry – i.e. when the stochastic clock reaches the barrier, the solution approaches the pay-off
in a smooth way. Concentrating on the $\sqrt{q}$ pay-off solutions, the general behaviour in the quadratic variation $q$ direction is preserved throughout time, so that a small $q$ initially will be cheaper than a large $q$. For pay-offs which depend on $X$ e.g. call option type $\max(X,0)$ we see in figure 5.4 the diffusive “Brownian” type behaviour we would expect for times less than expiry; note also in 5.4 similar diffusive behaviour for times less than expiry but with much less curvature – i.e. much less Brownian behaviour – this is because the surface was produced with a fixed lower quadratic variation. In (A.6) – payoff $\max(X,0)$ we have small instabilities when the quadratic variation is close to zero – this could be due to the lack of consistency between having a wide range of $X$ values and a small variance. It is also observed that more negative values of $\beta$ – the parameter in the CEV model, leads to flatter curvature in the initial solution, this is due to the damping nature the $\exp(\beta x)$ plays.

Figure 5.2: Solution of PDE (6.1) at $\tau = T$ with pay-off $\min(q, 10), \beta = -1/2$.

Figure 5.3: Solution of PDE (6.1) at $\tau = 0$ with pay-off $\min(q, 10), \beta = 0$. 
Figure 5.4: Solution of PDE (6.1) at \( \tau = T \) with pay-off \( \sqrt{q}, \beta = -1/2 \).

Figure 5.5: Solution of PDE (6.1) from \( \tau = 0 \) to \( \tau = T \) to with pay-off \( \sqrt{q}, \beta = -1/2 \).

Figure 5.6: Solution of PDE (6.1) from \( \tau = 0 \) to \( \tau = T \) to with pay-off (with High fixed Q) \( \max(X, 0), \beta = -1/2 \).
Figure 5.7: Solution of PDE (6.1) from $\tau = 0$ to $\tau = T$ to with pay-off (with Low fixed Q) $\max(X, 0), \beta = -1/2$. 
Chapter 6

Conclusion

In this thesis we have reviewed which products are actively traded on the market, and the common Derman model method to determine the fair price of volatility and variance swaps. We described and reviewed in greater depth the framework or Carr and Lee [20] on the pricing of options on quadratic variation with skew. We studied a key PDE in their work 6.1

\[ s^2(x) \left( \frac{1}{2} \frac{\partial^2 V^c(x, q, R)}{\partial x^2} - \frac{1}{2} \frac{\partial V^c(x, q, R)}{\partial x} + \frac{\partial V^c(x, q, R)}{\partial q} \right) = \frac{\partial V^c(x, q, R)}{\partial R}, \quad (6.1) \]

which gave us a necessary ingredients to price options contingent on the the quadratic variation when we knew the terminal value of the clock ex-ante. We solved this in the CEV case analytically

\[ V^c(x, q, R) = a_1 e^{x/2} I_\alpha \left( \frac{-\sqrt{2c_R e^{-\beta x}}}{\beta} \right) e^{c_R} e^{c_R R} + a_2 e^{x/2} K_\alpha \left( \frac{-\sqrt{2c_R e^{-\beta x}}}{\beta} \right) e^{c_R} e^{c_R R}. \quad (6.2) \]

and numerically by implementing an explicit finite difference scheme.

We didn’t discuss directly working in a two-factor model but instead used the concept of a stochastic clock (3.2) which our Future price Brownian motion ran on, to give the equivalence of stochastic volatility i.e.

\[ \sigma_t \equiv s \left( \log \frac{F_t}{F_0} \right) \alpha_t, \quad t \in [0, T] \quad (6.3) \]

to the model, and more – in our CEV example, we can really think of it as a SABR [70] example. Options on quadratic variation which expired when the stochastic clock reached a pre-determined barrier were then used to show how we could set up a hedge/replication for these options by trading in Bonds and dynamically in futures

\[ h(Y_S, \langle Y \rangle_S) = V^c(0, 0, H) + \int_0^S \frac{\partial V^c}{\partial x}(Y_t, \langle Y \rangle_t, H - \tau) \frac{dF_t}{F_t}. \quad (6.4) \]
In reality though we don’t know if a stochastic clock will reach a particular barrier level, so we also needed the probability distribution of the stochastic clock to know the distribution of possible pay-offs for all possible paths of the Futures. We showed how one could in principle determine this martingale representation of the stochastic clock and then use this to determine the full hedge/replication needed to price the options on quadratic variation – It was shown that for this, one needs to trade bonds, and dynamically in Futures and Options of many strikes

\[
h(Y_T, \langle Y \rangle_T) = V_0^h + \int_0^T \int_0^\infty dq_t(H) V^c(Y_t, \langle Y \rangle_t, H - \tau_t) dH dt
\]

\[
+ \int_0^T \int_0^\infty q_t(H) \frac{\partial}{\partial x} V^c(Y_t, \langle Y \rangle_t, H - \tau_t) dH \frac{1}{F_t} dF_t
\]

where

\[
V_0^h = \int_0^\infty q_0(H) V^c(0, 0, H) dH.
\]

A result which generalises the work of Derman et al.

To go forward, and continue this work, one could numerically determine the martingale representation of the stochastic clock, or equivalently its probability density function and integrate this against the analytical or numerical results we obtained to find obtain actual market prices of options on quadratic variation. It would also be interesting to look at what other classes of models the Carr and Lee framework captures by changing the skew function \(s(x)\).
A.1 Log contract Pricing – alternative derivation

For any twice-differentiable (pay-off) function \( g(S) \) and scalar \( c > 0 \), the following decomposition exists:

\[
g(S) = g(c) + g'(c)(S - c) + \int_c^\infty g''(K)(S - K)^+ dK + \int_0^c g''(K)(K - S)^+ dK. \tag{A.1}
\]

In the replicating context, the payoff of an arbitrary claim can be expressed as the payoff from \( g(c) \) bonds, \( g'(c) \) forward contracts with forward price \( c \), \( g''(K)dK \) calls struck above \( c \) and \( g''(c)dK \) puts struck below \( c \). Since we will make use of this, or similar results several times in this work, we outline the proof below. We assume \( S \) represents some underlying equity price and so is always positive. For any fixed \( c \),

\[
g(S) = g(c) + 1_{S>c} \int_c^S g'(u)du + 1_{S<c} \int_c^S g'(u)du
\]

\[
= g(c) + 1_{S>c} \int_c^S g'(u)du - 1_{S<c} \int_S^c g'(u)du
\]

\[
= g(c) + 1_{S>c} \int_S^c [g'(c) + \int_c^u g''(v)dv]du - 1_{S<c} \int_S^c [g'(c) - \int_u^c g''(v)dv]du. \tag{A.2}
\]

Where we have used the fundamental theorem of calculus, also note the use of the indicator function \( 1_{\text{condition}} \) is equal to 1 if the condition is true, and 0 otherwise. Fubini’s theorem now implies:

\[
g(S) = g(c) + g'(c)(S - c) + 1_{S>c} \int_c^S \int_v^S g''(v)dudv + 1_{S<c} \int_S^c \int_S^v g''(v)dudv. \tag{A.3}
\]

Finally, integrating over \( u \) finishes the proof:

\[
g(S) = g(c) + g'(c)(S - c) + 1_{S>c} \int_c^S \int_S^v g''(v)(S - v)dv + 1_{S<c} \int_S^c g''(v)(v - S)dv
\]

\[
= g(c) + g'(c)(S - c) + \int_c^\infty g''(v)(S - v)^+ dv + \int_0^c g''(v)(v - S)^+ dv. \tag{A.4}
\]

We note that we have not assumed any dynamics for the process, so this result is model-independent. We can use no-arbitrage to determine the initial price at \( t_0 \), \( V_{t_0}[g(S)] \) of \( g(S(T)) \) payable at maturity, time \( T \). We can see from (A.4) that \( V_{t_0}[g(S)] \) can be expressed in terms of the initial price \( \exp(-r(T - t_0)) \) of a bond maturing at \( T \) and the initial prices of calls and puts maturing at \( T \), and using put-call parity for the second term on the RHS of (A.4):

\[
V_{t_0}[g(S)] = g(c)\exp(-r(T - t_0)) + g'(c)[C(t_0, c) - P(t_0, c)]
\]

\[
+ \int_c^\infty g''(K)P(t_0, K)dK + \int_0^c g''(K)(t_0, K)dK. \tag{A.5}
\]
Setting $c = F_0 = S(t_0) \exp((r(T - t_0)))$ which is the forward stock price\(^1\), the initial price then can be written as\(^2\)

$$V_{t_0}[g(S)] = g(F_0) \exp(-r(T - t_0)) + \int_0^{F_0} g''(K)P(t_0, K)dK + \int_{F_0}^{\infty} g''(K)C(t_0, K)dK.$$  
(A.6)

This is a general formula for any twice-differentiable function $g(S)$, so we note that expression would not hold if we allowed for jumps in our model. If we apply the expression in (A.5) for the case $g(S) = \log(S)$, we obtain:

$$V_{t_0}[\log(S)] = \exp(-r(T - t_0))\log(c) + \frac{C(t_0, c) - P(t_0, c)}{c} - \int_0^{c} \frac{P(t_0, K)}{K^2}dK - \int_{c}^{\infty} \frac{C(t_0, K)}{K^2}dK.$$  
(A.7)

\(^1\)Excluding dividends; if we include discrete or continuous dividends, then the affect if that these will decrease the forward curve in the future since paying a positive dividend reduces the stock price.

\(^2\)The second term will vanish due to put-call parity $C(t_0, K) - P(t_0, K) = 0$
A.2 Matlab code for fair volatility/variance swap pricing

% Example using FTSE options on 02/04/2012.

% Choose the maturity for which to calculate the future vol (1: '04/20/2012'; 2:'06/15/2012'; 3: '12/21/2012'; 4: '06/21/2013'; 5:'12/20/2013').
SelectedMaturity = 5; %'12/20/2013'

% Auxiliary Function
intermediateCalc = @(Spot, SpotCutoff, T) (2/T)*((Spot - SpotCutoff)/SpotCutoff - log(Spot/SpotCutoff));

% Input the FTSE100 spot price on 04/06/2012 and define the cutoff
TodaysSpot = 5723.67; %Taken from Bloomberg
TodaysSpotCutoff = TodaysSpot;

% Input the implied volatility surface for FTSE100 on 02/04/2012, taken from Bloomberg
ImpliedVolatilityMatrix=[0.576320 0.487651 0.399766 0.311851 0.230544 0.172379 . . .
% We don’t have a strike exactly ATM but taking the 6th element is close
Strike=[4300 4600 4900 5200 5500 5800 6100 6400 6700 7000];

DateToday = '04/06/12';

Mat = {'04/20/2012'; '06/15/2012'; '12/21/2012'; '06/21/2013'; '12/20/2013'};
TimeInYears = (datenum(Mat) - datenum(DateToday))/365.0; %Act365 dayt count convention

% Input the spot rates based on the 06/04/2012 GBP Libor deposit/swap rates
YieldCurveDates ={'04/10/2012';'05/05/2012';'06/05/2012';'06/20/2012';'07/05/2012';...
Years = (datenum(YieldCurveDates) - datenum(DateToday))/365;
%See raw data in excel sheet VarianceSwapCalcInputData.xls
Rates=[0.010343474 0.010343474 0.010343474 0.010343474 0.010264309 0.010138318 0.010063389 . . .
% Extract the interest rate for the selected maturity
\[ T = \text{TimeInYears(SelectedMaturity)}; \]
\[ r = \text{interp1(Years, Rates, T)}; \]

%read the implied volatilities at the selected maturity
CallOptionV = \text{ImpliedVolatilityMatrix(SelectedMaturity, 6:10)};
P utOptionV = \text{ImpliedVolatilityMatrix(SelectedMaturity, 1:6)};
CallOptionK = \text{Strike(6:10)};
CallOptionKI = \{\text{CallOptionK(1)} + 0.1:0.1:\text{CallOptionK(end)}\};
CallOptionVI = \text{interp1(CallOptionK, CallOptionV, CallOptionKI, 'linear')};
PutOptionK = \text{Strike(1:6)};
PutOptionKI = \{\text{PutOptionK(1):0.1:PutOptionK(end)-0.1}\};
PutOptionVI = \text{interp1(PutOptionK, PutOptionV, PutOptionKI, 'linear')};
N = \text{length(CallOptionVI)};
K = \text{CallOptionKI};
V = \text{CallOptionVI};

%Numerical integration of Put and Call parts
\begin{verbatim}
for i=1:N-1
    intermediateCall(i) = (intermediateCalc(K(i+1), TodaysSpotCutoff, T)
                          - intermediateCalc(K(i), TodaysSpotCutoff, T)) / (K(i+1) - K(i));
    if i==1
        CallOptionFactor(1) = intermediateCall(1);
    end
    CallOptionValue(i) = \text{BSPrice(TodaysSpot, K(i), r, T, V(i), 'Call')};
    if i>1
        CallOptionFactor(i) = intermediateCall(i) - intermediateCall(i-1); end
    end
    CallOptionPart(i) = CallOptionValue(i) * CallOptionFactor(i);
end
CallOptionIntegral = \text{sum(CallOptionPart)};
\end{verbatim}

N = \text{length(PutOptionVI)};
K = \text{fliplr(PutOptionKI)};
V = \text{fliplr(PutOptionVI)};
\begin{verbatim}
for i=1:N-1
    intermediatePut(i) = (intermediateCalc(K(i+1), TodaysSpotCutoff, T)
                          - intermediateCalc(K(i), TodaysSpotCutoff, T)) / (K(i) - K(i+1));
    if i==1
        PutOptionFactor(1) = intermediatePut(1);
    end
    PutOptionValue(i) = \text{BSPrice(TodaysSpot, K(i), r, T, V(i), 'Put')};
    if i>1
        PutOptionFactor(i) = intermediatePut(i) - intermediatePut(i-1); end
    end
    PutOptionPart(i) = PutOptionValue(i) * PutOptionFactor(i);
end
PutOptionIntegral = \text{sum(PutOptionPart)};
\end{verbatim}

43
PutOptionValue(i) = BSPrice(TodaysSpot, K(i), r, T, V(i), 'Put');
if i>1
PutOptionFactor(i) = intermediatePut(i) - intermediatePut(i-1);
end
PutOptionPart(i) = PutOptionValue(i) * PutOptionFactor(i);
end
PutOptionIntegral = sum(PutOptionPart);
CallAndPutIntegrals = CallOptionIntegral + PutOptionIntegral;

% Estimate of fair variance
FairVariance = 2/T*(r*T - (TodaysSpot/TodaysSpotCutoff*exp(r*T) - 1) - log(TodaysSpotCutoff/TodaysSpot)) + exp(r*T)*CallAndPutIntegrals;

% Estimate of fair volatility
FaitVolatility = sqrt(FairVariance)*100;
A.3 Bessel Functions

Bessel functions $y(x)$ are solutions of Bessel’s differential equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$ \hspace{1cm} (A.8)

where $\alpha$ is an arbitrary real or complex number which is called the order of the Bessel function. The Bessel functions are divided into those of the first kind $J_\alpha(x)$ and second kind $Y_\alpha(x)$; these are linearly independent. We plot $J_\alpha(x)$ below\(^3\) for $\alpha = 0, 1, 2$

A.3.1 Modified Bessel functions

Modified Bessel functions $y(x)$ are solutions of the differential equation [1]:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \alpha^2)y = 0.$$ \hspace{1cm} (A.9)

The two sets of linearly independent solutions $I_\alpha(x)$ (of the first kind) and $K_\alpha(x)$ (second kind). These are related to the Bessel functions above by:

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + \alpha + 1)} \left( \frac{x}{2} \right)^{2m+\alpha}$$

$$K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha \pi)}.$$ \hspace{1cm} (A.10)

Plots of $I_\alpha(x)$ and $K_\alpha(x)$ are shown below: We can see that $I_\alpha(x)$ explodes for large $x$ and $K_\alpha(x)$ explodes for small $x$. This behaviour will be important when choosing between two different sets of solutions for our particular needs.

A.3.2 Modified Bessel Solutions

Plots of $e^{x/2} K_\alpha(e^{x/2})$ and $e^{x/2} I_\alpha(e^{x/2})$ showing how the solution behaves in the $x = \log(F(t)/F_0)$ variable.

\(^3\)This was produced in Maple: plot({BesselJ(0, x), BesselJ(1, x), BesselJ(2, x)}, x = 0 .. 100, color = [red, blue, green]);
Figure A.1: Modified Bessel function of the first kind, $I_\alpha(x)$

Figure A.2: Modified Bessel function of the second kind, $K_\alpha(x)$
Figure A.3: Modified Bessel function solution, $x$ dependence $e^{x/2}K_\alpha(e^{x/2})$.

Figure A.4: Modified Bessel function solution, $x$ dependence $e^{x/2}I_\alpha(e^{x/2})$. 
A.4 Toy examples

We can look at how (6.1) can be solved in simple circumstances. The PDE is linear in \( V \) and because of the presence of the second and first derivatives in \( x \), this rules out any possible dilation groups which could have been used to simplify the problem. Let’s assume at the start that \( s(x) \) is a constant, and \( wlog \ s(x) = 1 \) which could be achieved by rescaling the stochastic time coordinate of the problem. We then have:

\[
\frac{1}{2} \frac{\partial^2 V}{\partial x^2} - \frac{1}{2} \frac{\partial V}{\partial x} + \frac{\partial V}{\partial q} = \frac{\partial V}{\partial R} \tag{A.11}
\]

and we’ll assume further than the final condition \( h(x, q) \) depends only on \( q \), say \( h(x, q) = H(q) \) then there is no \( x \) dependence and the problem reduces to

\[
\frac{\partial V}{\partial q} = \frac{\partial V}{\partial R} \quad V(q, 0) = H(q) \tag{A.12}
\]

This has the simple solution \( V = H(q + R) \). We are more interested in the more complicated case when \( s(x) \) depends on \( x \) since this is a requirement to allow the solutions to capture the skew like behaviour. We can make the following observations to get insight into what more useful solutions could look like. If we write

\[
V(x, q, R) = \exp(\alpha x + \beta q) U(x, q, R), \tag{A.13}
\]

and now choosing particular \( \alpha \) and \( \beta \), the problem reduces to

\[
s(x)^2 \left( \frac{1}{2} \frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial q} \right) = \frac{\partial U}{\partial R} \quad U(x, q, 0) = f(x, q), \tag{A.14}
\]

where

\[
f(x, q) = e^{-(\alpha x + \beta q)} h(x, q). \tag{A.15}
\]

This problem is purely diffusive in \( x \), and leads one to look for other simplifications which can help in the solving of the problem e.g. \( s(\lambda x) = \lambda^a s(x) \) for constant \( \alpha \), then we can look for non-trivial dilation groups and hence some similarity reductions for the PDE, but achieving this in a way which is consistent with realistic financial
boundary conditions is not guaranteed to work. Expanding on this though and taking $s(x) = 1$, we then have\(^4\)

$$\frac{1}{2} \frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial q} = \frac{\partial U}{\partial R}$$

$$U(x, q, 0) = f(x, q). \quad (A.16)$$

$$U(x, q, 0) = f(x, q). \quad (A.17)$$

This can be solved by taking the Fourier transform \([1]\) in $x$, then use the method of characteristics to solve the first order PDE and then the convolution theorem to invert back the Fourier transform to obtain a solution

$$U(x, q, R) = \int_{-\infty}^{\infty} f(\psi, q + R)G(x - \psi, R)d\psi \quad (A.18)$$

where

$$G(z, R) = \frac{1}{\sqrt{2\pi R}}e^{-z^2/2R}. \quad (A.19)$$

This have a diffusive behaviour in $x$ and wave-like (hyperbolic) in $q$. This leads us to conclude that it wouldn’t be possible to find a general solution involving only the characteristic variable $\eta = R + q$ and $x$.

To progress from these ideas, we consider the skew function to have a more sophisticated but still simple form which will allow us to capture more of the skew we require. We will show in the bulk text that when $s(x)$ has a form consistent with CEV dynamics, that we can also find an analytical solution.

\(^4\)These ideas were brought to my attention by email conversations with Dr. Jeff Dewynne.
A.5 Explicit Finite Difference Matlab implementation of (6.1)

N = 30; % number of X steps
L = 30; % number of q steps
M = 3000; % number of time steps
dt = 1/M;
K = 10; % strike
F0 = 1; % initial Forward Price
Fmax = 1.4; % Max Expected Forward Price
Fmin = 0.6; % Max Expected Forward Price
Xmin = log(Fmin/F0);
Xmax = log(Fmax/F0);
dX = (Xmax - Xmin)/N;
Qmin = 0.01;
Qmax = 15.1;
dq = (Qmax - Qmin)/L;
X = linspace(Xmin, Xmax, N+1).
Q = linspace(Qmin, Qmax, L+1).
beta = -1.5;

A = 0.5*dt*exp(2*beta*X).*(1/dX).*(1/dX)-0.25*dt*exp(2*beta*X).*(1/dX);
B = 1-dt*exp(2*beta*X).*(1/dX).*(1/dX)+dt*exp(2*beta*X)/(dq);
C = 0.5*dt*exp(2*beta*X).*(1/dX).*(1/dX)+0.25*dt*exp(2*beta*X).*(1/dX);
D = -dt*exp(2*beta*X)/(dq);

Mat = zeros(N+1, L+1, M+1);

% upper boundary condition for max q boundary. One-sided ness due to the % hyperbolicity in q
for i = 1:N+1
  for j = 1:M+1
    Mat(i,L+1,j) = K;
  end
end
% pay-off condition
for i=1:N+1
    for l=1:L+1
        Mat(i,l,M+1)=min(Q(l),K);
    end
end

% boundary corrections for Xmin and Xmax to make solution more stable
for j=M+1:-1:2
    for l=1:L+1
        Mat(N+1,l,j-1)=Mat(N+1,l,j);
        Mat(1,l,j-1)=Mat(1,l,j);
    end
end

for j=M+1:-1:2
    for i=2:N
        for l=L+1:-1:2
            Mat(i,l,j-1)=A(i).*Mat(i+1,l,j)+B(i).*Mat(i,l,j)+C(i).*Mat(i-1,l,j)
                        +D(i).*Mat(i,l-1,j);
        end
    end
end

[Qc,Xc]=meshgrid(Q,X);
Z=Mat(:,:,1);
surf(Xc,Qc,real(Z));
xlabel('X');
ylabel('Q');
zlabel('Payoff')
A.6 Instability at small quadratic variation points

Figure A.5: Solution of PDE (6.1) at $\tau = 0$ with pay-off $\max(X, 0), \beta = -1/2$. 
A.7 Relation between density of process and option prices

Let $C(S_t, K, T - t)$ be the call option prices with the usual definitions for the parameters, ignoring any discounting. Also, let $\phi(S_T)|_K$ be the probability density that the stock price $S$ at time $T$ should be in the range $[K, K + \delta K]$, given that it has value $S_t$ at time $t$. By definition, we have

$$C(S_t, K, T - t) = \int_0^\infty (S_T - K)^+ \phi(S_T) dS_T = \int_K^\infty (S_T - K) \phi(S_T) dS_T.$$ 

Taking the derivative with respect to $K$, give us\textsuperscript{5}

$$\frac{\partial C(S_t, K, T - t)}{\partial K} = \frac{\partial}{\partial K} \int_K^\infty (S_T - K) \phi(S_t) dS_t$$

$$= -\frac{\partial}{\partial K} \int_0^K (S_T - K) \phi(S_T) dS_T$$

$$= -(K - K) - \int_K^\infty \frac{\partial [(S_T - K) \phi(S_T)]}{\partial K} dS_T$$

$$= - \int_K^\infty \phi(S_T) dS_T. \tag{A.20}$$

Differentiating again with respect to the strike gives us

$$\frac{\partial^2 C(S_t, K, T - t)}{\partial K^2} = \phi(K).$$

\textsuperscript{5}We use

$$\frac{\partial}{\partial x} \int_0^x f(t, x) dt = f(t, t) + \int_0^x \frac{\partial f(t, x)}{\partial x} dt$$
Bibliography


[33] Crosby, J Optimal hedging of variance derivatives. Submitted for publication 2010

[34] Crosby, J Variance derivatives: Pricing and convergence. This is joint work with Mark Davis of Imperial College London. Submitted for publication. 2011

[35] Variance swaps and volatility derivatives Lectures for the M.Sc. and Ph.D. courses in Quantitative Finance in the Department of Economics, Glasgow University.


[42] Emanuel, D, MacBeth, J.D. Further Results of the Constant Elasticity of Volatility Call Option Pricing Model. Journal of Financial and Quantitative Analysis, 4: 533553


[69] Finite Difference Methods I and II, Oxford University MSc lecture notes


[71] Rebonato, R Volatility and Correlation: The Perfect Hedger and the Fox (Wiley Finance)


[74] Kumar, V Indexes fall hard on bloody Friday, Market Watch, 2008